Fill Ups, of Definite Integrals and Applications

Q. 1. $f(x)=\begin{vmatrix} \sec x & \cos x & \sec^2 x+\cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$.

Then $\int_0^{\pi/2} f(x)\,dx = \ldots\ldots$

(1987 - 2 Marks)

Ans. $-\left(\frac{15\pi+32}{60}\right)$ **2.** $2-\sqrt{2}$

Solution. Given that,

$$f(x)=\begin{vmatrix} \sec x & \cos x & \sec^2 x+\cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Operating $R_1 - \sec x . R_3$,

$$=\begin{vmatrix} 0 & 0 & \sec^2 x+\cot x \operatorname{cosec} x-\cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Expanding along R_1, we get

$$=(\sec^2 x+\cot x \operatorname{cosec} x-\cos x)(\cos^4 x-\cos^2 x)$$
$$=\left(\frac{1}{\cos^2 x}+\frac{\cos x}{\sin^2 x}-\cos x\right)\cos^2 x(\cos^2 x-1)$$
$$=-\sin^2 x-\cos^5 x$$
$$\therefore \int_0^{\pi/2} f(x)dx=-\int_0^{\pi/2}(\sin^2 x+\cos^5 x)dx$$

Using

$$\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx=\frac{(n-1)(n-3)\ldots 2 \text{ or } 1}{(n)(n-2)\ldots 2}$$

Multiply the above by π/2 when n is even. We get

$$= -\left[\frac{1}{2}.\frac{\pi}{2}+\frac{4}{5}.\frac{2}{3}\right] = -\left[\frac{\pi}{4}+\frac{8}{15}\right] = -\left(\frac{15\pi+32}{60}\right)$$

Q. 2. The integral $\int_0^{1.5}[x^2]dx,$ **(1988 - 2 Marks)**

Where [ ] denotes the greatest integer function, equals ....

Ans. 2 - √2

Solution.

$\int_0^{1.5}[x^2]dx,$

We have $0 < x < 1.5 \Rightarrow 0 < x^2 < 2.25$

$\therefore [x^2] = 0, 0 < x^2 < 1 = 1, 1 \le x^2 < 2 = 2, 2 \le x^2 < (1.5)^2$

or $[x^2] = 0, 0 < x < 1 = 1, 1 \le x < \sqrt{2} = 2, \sqrt{2} \le x < 1.5$

$$\therefore \quad I = \int_0^{1.5}[x^2]dx = \int_0^1 0dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{1.5} 2dx$$

$$= 0 + [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{1.5}$$

$$= \sqrt{2} - 1 + 3 - 2\sqrt{2} = 2 - \sqrt{2}$$

Q. 3. The value of $\int_{-2}^{2}|1-x^2|\,dx$ is....... **(1989 - 2 Marks)**

Ans. 4

Solution.

Let $I = \int_{-2}^{2}|1-x^2|dx = 2\int_0^2|1-x^2|\,dx$

$$\left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f \text{ is an even function}\right]$$

$$= 2\int_0^1 (1-x^2)dx + 2\int_1^2 (x^2-1)dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^3}{3} - x\right]_1^2 \quad = \frac{4}{3} + \frac{8}{3} = \frac{12}{3} = 4$$

Q. 4. The value of $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi} d\phi$ is .... **(1993 - 2 Marks)**

Ans. $\pi(\sqrt{2}-1)$

Solution.

We have, $I = \int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi} d\phi \qquad \ldots(1)$

$$\Rightarrow \quad I = \int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin(\pi-\phi)} d\phi$$

$$\left[\text{Using} \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$$\Rightarrow \quad I = \int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin\phi} d\phi \qquad \ldots(2)$$

$$2I = \int_{\pi/4}^{3\pi/4} \frac{\pi}{1+\sin\phi} d\phi$$

$$= \pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{1-\sin^2\phi} d\phi = \pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{\cos^2\phi} d\phi$$

$$= \pi\int_{\pi/4}^{3\pi/4} (\sec^2\phi - \sec\phi\tan\phi)\, d\phi$$

$$= \pi[\tan\phi - \sec\phi]_{\pi/4}^{3\pi/4}$$

$$= \pi[\tan 3\pi/4 - \sec 3\pi/4 - \tan\pi/4 + \sec\pi/4]$$

$$= 2\pi(\sqrt{2}-1) \quad \Rightarrow \quad I = \pi(\sqrt{2}-1)$$

Q. 5. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$ is .... **(1994 - 2 Marks)**

Ans. ½

Solution.

$$\text{Let } I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx \qquad \ldots(1)$$

$$I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx \qquad \ldots(2)$$

$$\left[\text{Using} \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

Adding (1) and (2), we get

$$2I = \int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int_2^3 1dx = \frac{1}{2}(3-2) = \frac{1}{2}$$

Q. 6. If for nonzero $x,\ af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ where $a \neq b$, **then** $\int_1^2 f(x)\,dx = \ldots\ldots.$ **(1996 - 2 Marks)**

Ans. $\frac{1}{a^2-b^2}\left[a(\log 2 - 5) + \frac{7b}{2}\right]$

Solution. $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \qquad \ldots(1)$

Integrating both sides within the limits 1 to 2, we get

$$a\int_1^2 f(x)dx + b\int_1^2 f\left(\frac{1}{x}\right)dx = [\log x - 5x]_1^2 = \log 2 - 5 \ldots(2)$$

Replacing $\frac{1}{x}$ in (1), we get $af\left(\frac{1}{x}\right) + bf(x) = x - 5$

Integrating both sides within the limits 1 to 2, we get

$$a\int_1^2 f\left(\frac{1}{x}\right)dx + b\int_1^2 f(x)dx = \left[\frac{x^2}{2} - 5x\right]_1^2 = -\frac{7}{2} \ldots(3)$$

Eliminate $\int_1^2 f\left(\frac{1}{x}\right)$ between (2) and (3) by multiplying (2) by a and (3) by b and subtracting

$$\therefore \quad (a^2 - b^2)\int_1^2 f(x)dx = a(\log 2 - 5) + b.\frac{7}{2}$$

$$\therefore \quad \int_1^2 f(x)dx = \frac{1}{(a^2 - b^2)}\left[a(\log 2 - 5) + \frac{7b}{2}\right]$$

Q. 7. For $n > 0$, $\int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = ...$ **(1996 - 1 Mark)**

Ans. π^2

Solution.

Let $I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)} dx$$

$$\left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a - x)dx\right]$$

$$I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \qquad \ldots(2)$$

Adding (1) and (2) we get

$$2I = \int_0^{2\pi} \frac{2\pi \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow \quad I = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow \quad I = 2\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\left[\text{Using} \int_0^{2a} f(x)dx = 2\int_0^a (x)dx \text{ if } f(2a - x) = f(x)\right]$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \qquad \ldots (3)$$

[Using above property again]

$$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} dx \qquad ....(4)$$

$$\left[\text{Using} \int_0^a f(x)dx = \int_0^a (a-x)dx\right]$$

Adding (3) and (4) we get

$$2I = 4\pi\int_0^{\pi/2} 1.dx = 4\pi\left(\frac{\pi}{2} - 0\right) = 2\pi^2 \Rightarrow \text{I} = \pi^2$$

Q. 8. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \, In\, x)}{x} dx$ is ...... **(1997 - 2 Marks)**

Ans. 2

Solution.

$$\text{Let } I = \int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$$

Let $\pi \ln x = t$

$$\Rightarrow \quad \frac{\pi}{x} dx = dt \text{ also as } x \to 1, t \to 0, x \to e^{37}, t \to 37\pi$$

$$\therefore \quad I = \int_0^{37\pi} \sin t \, dt = [-\cos t]_0^{37\pi} = -\cos 37\pi + 1$$

$$= -(-1) + 1 = 2$$

Q. 9. Let $\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}, x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1)$ **then one of the possible values of k is ....** **(1997 - 2 Marks)**

Ans. 16

Solution.

$$\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1) = [F(x)]_1^k$$

Put $x^2 = t$

$\therefore \quad 2x dx = dt$; At $x=1, t=1$ and at $x=4, t=16$

$\therefore \quad I = \int_1^{16} \frac{e^{\sin t}}{t} dt = F[(t)]_1^{16} \qquad \therefore \ k = 16.$

True / False

Q. 1. The value of the integral $\int_0^{2a} [\frac{f(x)}{\{f(x)+f(2a-x)\}}]dx$ is **equal to a. (1997 - 2 Marks)**

Ans. T

Solution.

$$\text{Let } I = \int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)} dx \qquad \dots (1)$$

$$= \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f[2a-(2a-x)]} dx$$

[Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$$I = \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)} \qquad \dots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{2a} \frac{f(x)+f(2a-x)}{f(x)+f(2a-x)} dx = \int_0^{2a} 1.dx$$

$$= [x]_0^{2a} = 2a \Rightarrow I = a$$

∴ The given statement is true.

Subjective Problems of Definite Integrals & Applications (Part - 1)

Q. 1. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y – 2$. **(1981 - 4 Marks)**

Ans. $\frac{9}{8}$ sq. units

Solution. To find the area bounded by

$x^2 = 4y$ ..(1)

which is an upward parabola with vertex at (0, 0).

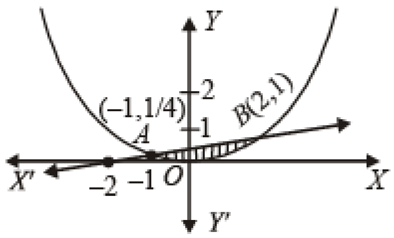
which is a st. line with its intercepts as –2 and 1/2 on axes. For Pt’s of intersection of

(1) and (2) putting value of 4y fom (2) in (1) we get

$x^2 = x+2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = 2,-1 \Rightarrow y = 1,1/4$

$\therefore A(-1,1/4)B(2,1)$.



Shaded region in the fig is the req area.

$\therefore$ Required area $= \int_{-1}^{2} (y_{line} - y_{parabola})dx$

$$= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx \quad = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(2+4-\frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right] = 9/8 \text{ sq. units}$$

Q. 2. Show that: $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{6n} \right) = \log 6$ **(1981 - 2 Marks)**

Solution. We know that in integration as a limit sum

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(r/n)$$

Similarly the given series can be written as

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{6n} \right) = \lim_{n\to\infty} \sum_{r=1}^{5n} \frac{1}{n+r}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{5n} \frac{1}{1+\frac{r}{n}}$$

$$= \int_0^5 \frac{1}{1+x} dx = [\log|1+x|]_0^5 = \log 6 - \log 1 = \log 6$$

Q. 3. Show that $\int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$. **(1982 - 2 Marks)**

Solution. Let $I = \int_0^{\pi} xf(\sin x)dx$ ... (1)

$$\Rightarrow \quad I = \int_0^{\pi} (\pi - x) f(\sin x)dx$$

Adding (1) and (2), we get, $2I = \int_0^{\pi} \pi f(\sin x)dx$

$$I=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

Hence Proved.

Q. 4. Find the value of $\int_{-1}^{3/2} |x\sin\pi x|\,dx$ **(1982 - 3 Marks)**

Ans. $\frac{3}{\pi}+\frac{1}{\pi^2}$

Solution.

$$\int_{-1}^{3/2} |x\sin\pi x|\,dx$$

For $-1\le x<0 \Rightarrow -\pi<px<0 \Rightarrow \sin\pi x<0$

$\Rightarrow x\sin\pi x>0$

For $1<x<3/2 \Rightarrow \pi<\pi x<3\pi/2 \Rightarrow \sin\pi x<0$

$\Rightarrow x\sin\pi x<0$

$$\therefore \int_{-1}^{3/2} |x\sin\pi x|\,dx=\int_{-1}^{1} x\sin\pi x\,dx+\int_{1}^{3/2}(-x\sin\pi x)ax$$

$$=2\int_0^1 x\sin\pi x\,dx-\int_1^{3/2} x\sin\pi x\,dx$$

$$=2\left[\frac{-x\cos\pi x}{\pi}+\frac{\sin\pi x}{\pi^2}\right]_0^1-\left[\frac{-x\cos\pi x}{\pi}+\frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

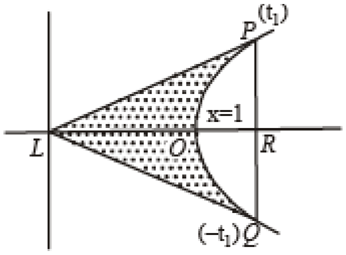
$$=2\left[\left(\frac{-\cos\pi}{\pi}+0\right)-(0+0)\right]$$

$$-\left[\left(\frac{-3/2\cos3\pi/2}{\pi}+\frac{\sin3\pi/2}{\pi^2}\right)-\left(\frac{-\cos\pi}{\pi}+\frac{\sin\pi}{\pi^2}\right)\right.$$

$$=2\left[\frac{1}{\pi}\right]-\left[-\frac{1}{\pi^2}-\frac{1}{\pi}\right]=\frac{2}{\pi}+\frac{1}{\pi^2}+\frac{1}{\pi}\ =\frac{3}{\pi}+\frac{1}{\pi^2}$$

Q. 5. For any real t, $x=\frac{e^t+e^{-t}}{2}$, $y=\frac{e^t-e^{-t}}{2}$ is **is a point on the hyperbola $x^2-y^2=1$. Show that the area bounded by this hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1.** **(1982 - 3 Marks)**

Solution. Let P(t_1) and Q($-t_1$) be two points on the hyperbola.



$$\text{Area (PRQOP)} = \int_{-t_1}^{t_1} y\,dx = \int_{-t_1}^{t_1} \left(\frac{e^t + e^{-t}}{2}\right)\left(\frac{dx}{dt}\right) dt$$

$$= \int_{-t_1}^{t_1} \left(\frac{e^t - e^{-t}}{2}\right) \frac{d}{dt}\left(\frac{e^t + e^{-t}}{2}\right) .dt$$

$$= \int_{-t_1}^{t_1} \left(\frac{e^t - e^{-t}}{2}\right) dt = \int_{-t_1}^{t_1} \frac{e^{2t} + e^{-2t} - 2}{4} dt$$

$$= \left[\frac{e2t}{8} - \frac{e^{-2t}}{8} - \frac{2t}{4}\right]_{-t_1}^{t_1} = \frac{2}{8}(e^{2t_1} - e^{-2t_1} - 4t_1)$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - t_1$$

$$\text{Area of } \Delta LPR = \frac{1}{2} LR \times PQ = LR \times PR$$

$$= \frac{e^{t_1} + e^{-t_1}}{2} \times \frac{e^{t_1} - e^{-t_1}}{2} = \frac{e^{2t_1} - e^{-2t_1}}{4} \qquad \ldots(2)$$

∴ The required area = Ar (ΔLPQ)- Ar (PRQOP)

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{e^{2t_1} - e^{-2t_1}}{4} + t_1 = t_1$$

Q. 6. Evaluate : $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$ **(1983 - 3 Marks)**

Ans. $\frac{1}{20}\log 3$

Solution.

$$I=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$$

Let sin x - cox x = t ⇒ as x ⟶ 0,t ⟶-1 as x ⟶ π / 4,t ⟶ 0

⇒ (cos x + sin x)dx= dt

Also, $t^2=1-\sin 2x \Rightarrow \sin 2x=1-t^2$

$$I=\int_{-1}^{0}\frac{dt}{9+16(1-t^2)}=\int_{-1}^{0}\frac{dt}{25-16t^2}$$

$$=\frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}=\frac{1}{16}\cdot\frac{1}{2.\frac{5}{4}}\log\left[\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$$

$$=\frac{1}{40}\left[\log 1-\log\frac{1}{9}\right]=\frac{\log 9}{40}=\frac{2\log 3}{40}=\frac{1}{20}\log 3$$

Q. 7. Find the area bounded by the x-axis, part of the curve $y=\left(1+\frac{8}{x^2}\right)$ **and the ordinates at x = 2 and x = 4. If the ordinate at x = a divides the area into two equal parts, find a. (1983 - 3 Marks)**

Ans. a = 2√2

Solution.

$$y=1+\frac{8}{x^2}$$

$$\text{Req. area}=\int_2^4 y\,dx=\int_2^4\left(1+\frac{8}{x^2}\right)dx=\left[x-\frac{8}{x}\right]_2^4=4$$

If x = 4a bisects the area then we have

$$\int_2^a \left(1+\frac{8}{x^2}\right)dx = \left[x-\frac{8}{x}\right]_2^a = \left[a-\frac{8}{a}-2+4\right] = \frac{4}{2}$$

$$\Rightarrow \quad a-\frac{8}{a}=0 \Rightarrow a^2=0 \quad \Rightarrow \quad a=\pm 2\sqrt{2}$$

$$\text{Since } 2<a<4 \qquad \therefore a = 2\sqrt{2}$$

Q. 8. Evaluate the following $\int_0^{\frac{1}{2}} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ **(1984 - 2 Marks)**

Ans. $\frac{6-\pi\sqrt{3}}{12}$

Solution.

$$\text{Let } I = \int_0^{1/2} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

Put x = sinθ ⇒ dx =cosθ d θ

Also when x = 0, θ = 0

and when x = 1/2, θ = θ / 6

$$\text{Thus, } I = \int_0^{\pi/6} \frac{\sin\theta\sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$$

$$\Rightarrow \quad I = \int_0^{\pi/6} \theta\sin\theta\, d\theta$$

Integrating the above by parts, we get

$$I = [\theta(-\cos\theta)]_0^{\pi/6} + \int_0^{\pi/6} 1.\cos\theta d\theta$$

$$= [-\theta\cos\theta + \sin\theta]_0^{\pi/6} = \frac{-\pi}{6}.\frac{\sqrt{3}}{2}+\frac{1}{2} = \frac{6-\pi\sqrt{3}}{12}$$

Q. 9. Find the area of the region bounded by the x-axis and the curves defined by (1984 - 4 Marks)

$$y = \tan x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}; \; y = \cot x, \quad \frac{\pi}{6} \le x \le \frac{3\pi}{2}$$

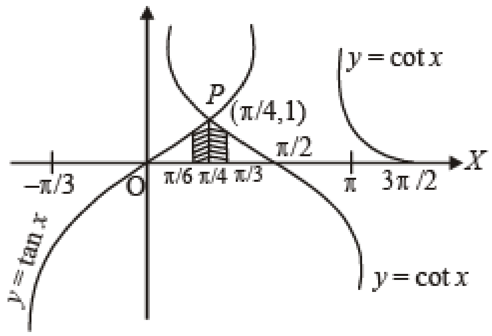
Ans. $\log\frac{3}{2}$ sq. units

Solution. To find the area bold by x - axis and curves

y = tan x, $-\pi/3 \le x \le \pi/3$ … (1)

and y = cot x, $\pi/6 \le x \le 3\pi/2$ … (2)

The curves intersect at P, where tan x = cot x, which is satisfied at x = π /4 within the given domain of x.



The required area is shaded area

$$A = \int_{\pi/6}^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/3} \cot x\,dx$$

$$= [\log \sec x]_{\pi/6}^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/3}$$

$$= \left(\log\sqrt{2} - \log\frac{2}{\sqrt{3}}\right) + \left(\log\frac{\sqrt{3}}{2} - \log\frac{1}{\sqrt{2}}\right)$$

$$= 2\left(\log\sqrt{2}.\frac{\sqrt{3}}{2}\right) = 2\log\sqrt{\frac{3}{2}} = \log 3/2 \text{ sq. units}$$

Q. 10. Given a function f(x) such that (1984 - 4 Marks)
(i) it is integrable over every interval on the real line and

(ii) f (t + x) = f (x), for every x and a real t, then show that the integral $\int_a^{a+t} f(x)\,dx$ is independent of a.

Solution.

$$\text{Let } \int f(x)dx = F(x)+c$$

$$\text{Then } F'(x) = f(x) \qquad \ldots(1)$$

$$\text{Now } I = \int_a^{a+t} f(x)dx = F(a+t)-F(a)$$

$$\therefore \quad \frac{dI}{da} = F'(a+t)-F(a) = f(a+t)-f(a)$$

[Using eq. (1)]

$$= f(a)-f(a) \quad \text{[Using given condition]}$$

$$=0$$

This shows that I is independent of a.

Q. 11. Evaluate the following : $\int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ **(1985 - 2½ Marks)**

Ans. $\frac{\pi^2}{16}$

Solution.

$$\text{Let } I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx \qquad \ldots(1)$$

$$I = \int_0^{\pi/2} \frac{(\pi/2-x)\sin(\pi/2-x)\cos(\pi/2-x)}{\cos^4(\pi/2-x)+\sin^4(\pi/2-x)}dx$$

$$\left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{(\pi/2 - x)\sin x \cos x}{\sin^4 x + \cos^4 x} dx \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\sec^2 x \tan x}{\tan^4 x + 1} dx$$

$$= \frac{\pi}{2 \times 4} \int_0^{\pi/2} \frac{2 \tan x \sec^2 x \, dx}{1 + (\tan^2 x)^2}$$

Put $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x \, dx = dt$

Also as $x \to 0, t \to 0$; as $x \to \pi/2, t \to \infty$

$$\therefore \quad I = \frac{\pi}{8} \int_0^{\infty} \frac{dt}{1 + t^2} = \frac{\pi}{8} [\tan^{-1} t]_0^{\infty} = \frac{\pi}{8} [\pi/2 - 0] = \pi^2/16$$

Q. 12. Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and y =| x - 1 | and find its area. (1985 - 5 Marks)

Ans. $\frac{5\pi - 2}{4}$ sq. units

Solution. The given curves are

$$y = \sqrt{5 - x^2} \qquad \ldots(1)$$

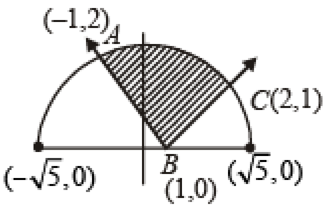
y =|x- 1| … (2)

We can clearly see that (on squaring both sides of (1)) eq. (1) represents a circle. But as y is + ve sq. root, ∴ (1) represents upper half of circle with centre (0, 0) and radius √5.

Eq. (2) represents the curve

$$y = \begin{cases} -x + 1 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 1 \end{cases}$$

Graph of these curves are as shown in figure with point of intersection of $y=\sqrt{5-x^2}$ and $y=-x+1$ as $A(-1,2)$ and of $y=\sqrt{5-x^2}$ and $y=x-1$ as $C(2,1)$



The required area = Shaded area

$$=\int_{-1}^{2}(y_{(1)}-y_{(2)})dx=\int_{-1}^{2}\sqrt{5-x^2}\,dx-\int_{-1}^{2}|x-1|\,dx$$

$$=\left[\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2}-\int_{1}^{1}\{-(x-1)\}\,dx-\int_{1}^{2}(x-1)dx$$

$$=\left(\frac{2}{2}\sqrt{5-4}+\frac{5}{4}\sin^{-1}\frac{2}{\sqrt{5}}\right)-\left(\frac{-1}{2}\sqrt{5-1}+\frac{5}{2}\sin{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)$$

$$-\left(\frac{-x^2}{2}+x\right)_{-1}^{1}-\left(\frac{x^2}{2}-x\right)_{1}^{2}$$

$$=1+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}+1+\frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)-\left[\left(\frac{-1}{2}+1\right)-\left(\frac{-1}{2}-1\right)\right]-\left[(2-2)-\left(\frac{1}{2}-1\right)\right]$$

$$=2+\frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{5}}\right]-2-\frac{1}{2}$$

$$=\frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}\frac{2}{\sqrt{5}}\right]-\frac{1}{2}=\frac{5}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}$$

$$=\frac{5\pi-2}{4}\text{ square units.}$$

Q. 13. Evaluate: $\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x},\ 0<\alpha<\pi$ **(1986 - 2½ Marks)**

Ans. $\frac{\pi\alpha}{\sin\alpha}$

Solution.

Let $I=\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}$ ...(1)

$I=\int_0^{\pi}\frac{(\pi-x)dx}{1+\cos\alpha(\sin(\pi-x))}$

[Using $\int_0^{a} f(x)dx=\int_0^{a} f(a-x)dx$]

$\therefore\quad I=\int_0^{\pi}\frac{(\pi-x)dx}{1+\cos\alpha\sin x}$ ...(2)

Adding (1) and (2), we get

$$2I=\int_0^{\pi}\frac{x+\pi-x}{1+\cos\alpha\sin x}dx=\int_0^{\pi}\frac{\pi}{1+\cos\alpha\sin x}dx$$

$$\therefore\quad I=\frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\cos\alpha\sin x}dx=\frac{\pi}{2}.2\int_0^{\pi/2}\frac{1}{1+\cos\alpha\sin x}dx$$

$$=\pi\int_0^{\pi/2}\frac{1}{1+\cos\alpha.\frac{2\tan x/2}{1+\tan^2 x/2}}dx$$

$$=\pi\int_0^{\pi/2}\frac{\sec^2}{1+\tan^2 x/2+2\cos\alpha\tan x/2}dx$$

Put $\tan x/2=t$, $\frac{1}{2}\sec^2\frac{x}{2}dx=dt \Rightarrow \sec^2 x/2dx=2dt$

Also when $x\to 0, t\to 0$ as $x\to\pi 2, t\to 1$

$$\therefore\quad I=\pi\int_0^{1}\frac{2dt}{t^2+(2\cos\alpha)t+1}$$

$$= 2\pi \int_0^1 \frac{dt}{(t+\cos\alpha)^2 + 1 - \cos^2\alpha} \quad = 2\pi \int_0^1 \frac{dt}{(t+\cos\alpha)^2 + \sin^2\alpha}$$

$$= 2\pi . \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^1$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{2\cos^2\alpha/2}{2\sin\alpha/2\cos\alpha/2}\right) - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}(\cot\alpha/2) - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}(\tan^{-1}(\pi/2-\alpha/2)) - \tan^{-1}(\tan(\pi/2-\alpha))\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2} + \alpha\right] = \frac{2\pi}{\sin\alpha}\left[\frac{\alpha}{2}\right] = \frac{\pi\alpha}{\sin\alpha}$$

Q. 14. Find the area bounded by th e curves, $x^2 + y^2 = 25$, $4y = | 4 – x^2 |$ and $x = 0$ above the x-axis. (1987 - 6 Marks)

Ans. $4 + 25\sin^{-1}\frac{4}{5}$

Solution. We have to find the area bounded by the curves

$x^2 + y^2 = 25$ ...(1)

$4y = |4 - x^2|$ ...(2)

$x = 0$ ...(3)

and above x-axis.

$$\text{Now, } 4y = |4x - x^2| = \begin{cases} 4 - x^2, \text{if } x^2 < 4 \\ x^2 - 4, \text{if } x^2 \geq 4 \end{cases}$$

$$4y = \begin{cases} 4 - x^2, \text{if } -2 < x < 2 \\ x^2 - 4, \text{if } x \geq 2 \text{ or } \leq -2 \end{cases}$$

Thus we have three curves

(I) Circle $x^2 + y^2 = 25$

(II) P_1 : Parabola, $x^2 = -4(y-1), -2 \le x \le 2$

(III) P_2 : Parabola, $x^2 = 4(y+1), x \ge 2 \text{ or } x \le -2$

(I) and (II) intersect at $-4y + 4 + y^2 = 25$

or $(y - 2)^2 = 5^2$ ∴ $y - 2 = \pm 5$ $y = 7, y = -3$

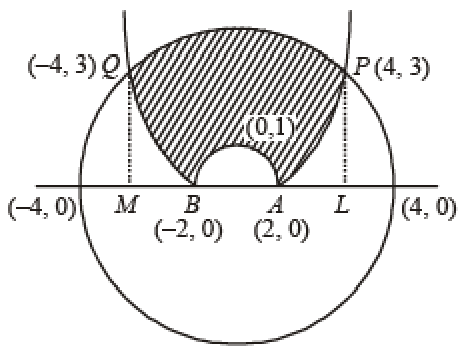
y = – 3, 7 are rejected since.

y = – 3 is below x-axis and

y = 7 gives imaginary value of x. So, (I) and (II) do not intersect but II intersects x-axis at (2, 0) and (–2, 0). (I) and (III) intersect at

$$4y + 4 + y^2 = 25 \quad \text{or} \quad (y+2)^2 = 5^2$$
$$\therefore \quad y + 2 = \pm 5 \qquad \therefore \quad y = 3, -7.$$

y = – 7 is rejected, y = 3 gives the points above x-axis. When y = 3, x = ±4. Hence the points of intersection of (I) and (III) are (4, 3) and (– 4, 3). Thus we have the shape of the curve as given in figure



Required area is

$$= 2\left[\int_0^4 y_{circle}\,dx - \int_0^2 y_{P_1}\,dx - \int_2^4 y_{P_2}\,dx\right]$$

$$= 2\left[\int_0^4 \sqrt{25-x^2}\,dx - \frac{1}{4}\int_0^2 (4-x^2)\,dx - \frac{1}{4}\int_0^4 (x^2-4)\,dx\right]$$

$$= 2\left[\left[\frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_0^4 \quad -\frac{1}{4}\left(4x - \frac{x^3}{3}\right)_0^2 - \frac{1}{4}\left(\frac{x^3}{3} - 4x\right)_2^4\right.$$

$$= 2\left[6 + \frac{25}{2}\sin^{-1}\frac{4}{5} - \frac{4}{3} - \frac{4}{3} - \frac{4}{3}\right]$$

$$= 12 + 25\sin^{-1}\frac{4}{5} - 8 = 4 + 25\sin^{-1}\frac{4}{5}$$

Q. 15. Find the area of the region bounded by the curve C : y = tan x, tangent drawn to C at x = π/4 and the x-axis. (1988 - 5 Marks)

Ans. $\frac{1}{2}\left[\log 2 - \frac{1}{2}\right]$ sq. units

Solution. The given curve is y = tan x ...(1)

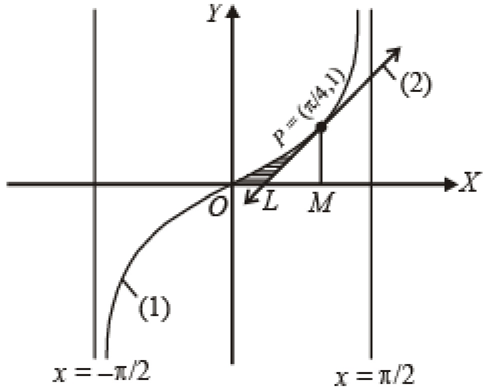
Let P be the point on (1) where x = π /4

∴ y = tan p / 4=1 i.e. co-ordinates of P are (π / 4,1)

∴ Equation of tangent at P is y - 1 = 2(x -π / 4)

or y = 2x + 1 -π/2 ...(2)

The graph of (1) and (2) are as shown in the figure.



Tangent (2) meets x-axis at, $L\left(\frac{\pi-2}{4}, 0\right)$

Now the required area = shaded area

= Area OPMO - Ar (ΔPLM)

$$= \int_0^{\pi/4} \tan x\, dx - \frac{1}{2}(OM - OL)PM$$

$$= [\log \sec x]_0^{\pi/4} - \frac{1}{2}\left\{\frac{\pi}{4} - \frac{\pi-2}{4}\right\}.1 \;= \frac{1}{2}\left[\log 2 - \frac{1}{2}\right] \text{ sq.units.}$$

Q. 16. Evaluate $\int_0^1 \log[\sqrt{1-x} + \sqrt{1+x}]dx$ **(1988 - 5 Marks)**

Ans. $\frac{1}{2}\left[\log 2 + \frac{\pi}{2} - 1\right]$

Solution.

Let $I = \int_0^1 1.\log[\sqrt{1-x} + \sqrt{1+x}]dx$

Intergrating by parts, we get

$$I = [x\log(\sqrt{1-x} + \sqrt{1+x})]_0^1$$

$$-\int_0^1 x.\frac{1}{\sqrt{1-x}+\sqrt{1+x}}.\left[\frac{-1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}}\right]dx$$

$$= \log\sqrt{2} - \int_0^1 x\frac{(\sqrt{1+x}-\sqrt{1-x})}{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}-\sqrt{1-x})} \cdot \frac{(\sqrt{1-x}-\sqrt{1+x})}{2\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\int_0^1 \frac{x(\sqrt{1+x}-\sqrt{1-x})^2}{(1+x-1+x)\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\int_0^1 \frac{1+x+1-x-2\sqrt{1-x^2}}{2\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}}dx - \frac{1}{2}\int_0^1 1dx$$

$$= \frac{1}{2}\left[\log 2 + \left(\sin^{-1}x\right)_0^1 - (x)_0^1\right] \;= \frac{1}{2}[\log 2 + \pi/2 - 1]$$

Subjective Problems of Definite Integrals & Applications (Part - 2)

Q. 17. If f and g are continuous function on [0, a] satisfying f(x) = f(a – x) and g(x) + g(a – x) = 2, then show that (1989 - 4 Marks)

Solution.

Let $I = \int_0^a f(x)g(x)dx = \int_0^a f(a-x)g(a-x)dx$

[Using the prop. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$= \int_0^a f(x)(2-g(x))dx$

As given that $f(a-x) = f(x)$ and $g(a-x)+g(x) = 2$

$= 2\int_0^a f(x)dx - \int_0^a f(x)g(x)dx$; $\quad \therefore \; I = 2\int_0^a f(x)dx - I$

$\Rightarrow \; 2I = 2\int_0^a f(x)dx \Rightarrow I = \int_0^a f(x)dx$

Hence the result.

Q. 18. Show that $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$ **(1990 - 4 Marks)**

Solution.

We have, $I = \int_0^{\pi/2} f(\sin 2x)\cos x dx$ ...(1)

$I = \int_0^{\pi/2} f(\sin 2x)\sin x dx$ ...(2)

[Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)(\cos x + \sin x)dx$$

$$\Rightarrow \quad 2I = 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$$

[Using the property,

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ when } f(2a-x) = f(x)]$$

$$\Rightarrow \quad I = \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$$

$$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin(\pi/4 + x)dx$$

$$= \sqrt{2}\int_0^{\pi/4} f\left[\sin\left(2\left(\frac{\pi}{4} - x\right)\right)\right]\sin(\pi/4 + \pi/4 - x)dx$$

$$\left[\begin{array}{l}\text{Using the property} \\ \int_0^a f(x)dx = \int_0^a f(a-x)dx\end{array}\right]$$

$$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x dx$$

Hence Proved.

Q. 19. Prove that for any positive integer k, $\frac{\sin 2kx}{\sin x} =$ **2[cos x + cos 3x + ... + cos (2k – 1) x] (1990 - 4 Marks)**

Hence prove that $\int_0^{\pi/2} \sin 2kx \cot x \, dx = \frac{\pi}{2}$

Solution. To prove : $\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + ... + \cos(2k-1)x]$

It is equivalent to prove that

sin 2kx = 2 sin x cos x + 2 cos 3x sinx+ ... + 2 cos(2k- 1) x sinx

Now, R.H.S. = (sin2 x) + (sin4 x- sin2x) + (sin6 x- sin4x) +.... +(sin 2kx - sin(2k- 2)x)

= sin2kx = L.H.S. Hence Proved.

$$\text{Now} \int_0^{\pi/2} \sin 2kx \cot x dx = \int_0^{\pi/2} \frac{\sin 2kx}{\sin x} . \cos x dx$$

$$= \int_0^{\pi/2} 2(\cos x + \cos 3x + ... + \cos(2k-1)x) \cos x dx$$

[Using the identity proved above]

$$= \int_0^{\pi/2} [2\cos^2 x + 2\cos 3x \cos x + 2\cos 5x \cos x + ... +2 \cos(2k-1) x \cos x] dx$$

$$= \int_0^{\pi/2} [(1+\cos 2x) + (\cos 4x + \cos 2x) +(\cos 6x + \cos 4x) + (\cos 2kx) + \cos(2k-2) x] dx$$

$$= \int_0^{\pi/2} 1 + 2[\cos 2x + \cos 4x + \cos 6x + ... \cos(2k-2)x] + \cos 2kx \, dx$$

$$= \left[x + 2\left\{ \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + ... + \frac{\sin(2k-2)x}{2k-2} \right\} + \frac{\sin 2kx}{2k} \right]_0^{\pi/2}$$

$= \pi /2 \quad [\because \sin n\pi = 0, \forall n \in N]$

Hence Proved

Q. 20. Compute the area of the region bounded by the curves y = ex In x

and $y = \frac{In\, x}{ex}$ where $\ln e = 1$. **(1990 - 4 Marks)**

Ans. $\frac{e^2-5}{4e}$

Solution. The given curves are

y = ex logex … (1)

and $y=\frac{\log_e x}{ex}$ … (2)

The two curves intersect where ex log x $x=\frac{\log x}{ex}$

$$\Rightarrow \left(ex-\frac{1}{ex}\right)\log x=0 \Rightarrow x=\frac{1}{e} \text{ or } x=1$$

$$\text{At } x=1/e \text{ or } ex=1, \log x=-\log e=-1, y=-1$$

At x = 1/e or ex = 1, log x = - log e =-1,y =-1

So that $\left(\frac{1}{e},-1\right)$ is one point of intersection and at x = 1,

log 1 = 0 ∴ y = 0

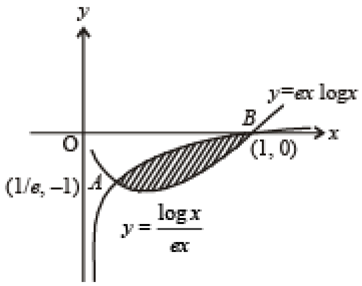
∴ (1, 0) is the other common point of intersection of the curves. Now in between these two points, $\frac{1}{e}<x<1$

$$\text{or } \log\left(\frac{1}{e}\right)<\log x<\log 1, \text{ or } -1<\log x<0$$

i.e. log x is – ve, throughout

$$\therefore \quad y_1=ex\log_e x, y_2=\frac{\log_e x}{ex}$$

Clearly under the condition stated above $y_1 < y_2$ both being –ve in the interval $\frac{1}{e}<x<1$. The rough sketch of the two curves is as shown in fig. and shaded area is the required area.



∴ The required area = shaded area

$$= \left|\int_{1/e}^{1}(y_1 - y_2)\,dx\right| = \left|\int_{1/e}^{1}\left[ex\log x - \frac{\log x}{ex}\right]dx\right|$$

$$= \left|e\int_{1/e}^{1}x\log x - \frac{1}{e}\int_{1/e}^{1}\frac{\log x}{x}dx\right|$$

$$= \left|e\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_{1/e}^{1} - \frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1}\right|$$

$$= \left|e\left[\left(-\frac{1}{4}\right) - \left(-\frac{1}{2e^2} - \frac{1}{4e^2}\right)\right] - \frac{1}{e}\left[0 - \frac{1}{2}\right]\right|$$

$$= \left|e\left[-\frac{1}{4} + \frac{3}{4e^2}\right] + \frac{1}{2e}\right| = \left|\frac{-e}{4} + \frac{3}{4e} + \frac{1}{2e}\right|$$

$$= \left|\frac{5}{4e} - \frac{e}{4}\right| = \left|\frac{5 - e^2}{4e}\right| = \frac{e^2 - 5}{4e}$$

Q. 21. Sketch the curves and identify the region bounded by x = 1/2, x = 2, y = In x and y = 2^x. Find the area of this region. (1991 - 4 Marks)

Ans. $\frac{4-\sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}$

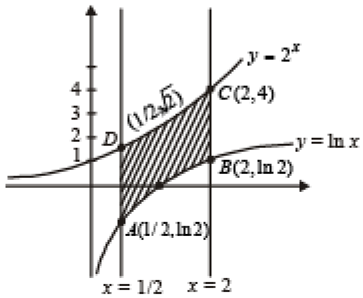
Solution. The given curves are x = 1/2 ....(1), x = 2...(2), y = ln x...(3),y = 2^x....(4)

Clearly (1) and (2) represent straight lines parallel to y - axis at distances 1/2 and 2

units from it, respectively. Line x =

1/2 meets (3) at $\left(\frac{1}{2}, -\ln 2\right)$ and (4) at $\left(\frac{1}{2}, \sqrt{2}\right)$. Line $x = 2$ meets (3) at (2, ln 2) and (4) at (2, 4).

The graph of curves are as shown in the figure.



Required area = ABCDA

$$= \int_{1/2}^{1} (-\ln x)dx + \int_{1/2}^{2} 2^x dx - \int_{1}^{2} \ln x\, dx$$

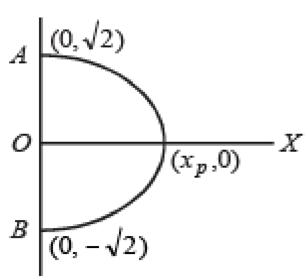
$$= \int_{1/2}^{2} 2^x dx - \int_{1/2}^{2} \ln x\, dx = \int_{1/2}^{2} (2^x - \ln x)\, dx$$

$$= \left[\frac{2^x}{\log 2} - (x \log x - x)\right]_{1/2}^{2}$$

$$= \left(\frac{4}{\log 2} - 2\log 2 + 2\right) - \left(\frac{\sqrt{2}}{\log 2} + \frac{1}{2}\log 2 - \frac{1}{2}\right)$$

$$= \left(\frac{4-\sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right)$$

Q. 22. If 'f' is a continuous function with $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$, **then show that every line y = mx**



intersects the curve $y^2 + \int_0^x f(t)\,dt = 2!$ **(1991 - 4 Marks)**

Solution. We are given that f is a continuous function and

$$\int_0^x f(t)dt \to \infty \text{ as} |x| \to \infty$$

To show that ever y line y = mx intersects the curve

$$y^2 + \int_0^x f(t)dt = 2.$$

If possible, let y = mx intersects the given curve, then Substituting y = mx in the equation of the curve we get

$$m^2x^2 + \int_0^x f(t)dt = 2 \qquad \text{............(1)}$$

Consider $F(x) = m^2x^2 + \int_0^x f(t)dt - 2$

Then F (x) is a continuous function as f (x) is given to be continuous.

Also F (x) → ∞ as |x|→ ∞

But F (0) = –2

Thus F (0) = –ve and F(b) = +ve where b is some value of x, and F (x) is continuous.

Therefore F (x) = 0 for some value of x Î (0,b) or eq. (1) is solvable for x.

Hence y = mx intersects the given curve.

Q. 23. Evaluate $\int_0^\pi \frac{x \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx$ **(1991 - 4 Marks)**

Ans. $\frac{8}{\pi^2}$

Solution.

Let $I = \int_0^\pi \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx$

Consider, 2x - p = y n ⇒ $dx = \frac{dy}{2}$, Also, $x = \left(\frac{\pi}{2} + \frac{y}{2}\right)$

When x → 0,y → -π when x → π ,y → π

∴ We get

$$I = \int_{-\pi}^{\pi} \frac{\left(\frac{\pi + y}{2}\right) \sin(\pi + y) \sin\left[\frac{\pi}{2}\cos\left(\frac{\pi}{2} + \frac{y}{2}\right)\right]}{y} \cdot \frac{dy}{2}$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} \left(\frac{\pi}{y} + 1\right)(-\sin y)\sin\left(\frac{-\pi}{2}\sin\frac{y}{2}\right) dy$$

$$= \frac{\pi}{4}\int_{-\pi}^{\pi} \frac{\sin y \sin(\pi/2 \sin y/2)}{y} dy \quad + \frac{1}{4}\int_{-\pi}^{\pi} \sin y \sin\left(\frac{\pi}{2}\sin\frac{y}{2}\right) dy$$

$$= 0 + \frac{2}{4}\int_{0}^{\pi} \sin y \sin(\pi/2 \sin y/2)\, dy$$

[Using $\int_{-a}^{a} f(x)\,dx = 0$ if f is odd function

$= 2\int_0^a f(x)\,dx$ if f is an even function]

$$\therefore I = \frac{1}{2}\int_0^{\pi} 2\sin y/2 \cos y/2 \sin(\pi/2 \sin y/2)\, dy$$

Let $\sin y/2 = u \Rightarrow \frac{1}{2}\cos y/2\, dy = du$

⇒ cos y / 2 dy = 2du

Also as y → 0,u → 0 and as y → π,u → 1

$$\therefore I = \int_0^1 2u \sin\left(\frac{\pi u}{2}\right) du$$

$$= \left[2u \frac{-\cos\frac{\pi u}{2}}{\pi/2}\right]_0^1 + \int_0^1 2 . \frac{2}{\pi}\cos\left(\frac{\pi u}{2}\right) du$$

$$= 0 + \frac{4}{\pi} \left. \frac{\sin\left(\frac{\pi u}{2}\right)}{\pi/2} \right|_0^1 = \frac{8}{\pi^2}\left(\sin\frac{\pi}{2} - 0\right) = \frac{8}{\pi^2}$$

Q. 24. Sketch the region bounded by the curves y = x^2 and $y = \frac{2}{1+x^2}$. **Find the area.** **(1992 - 4 Marks)**

Ans. $\left(\pi - \frac{2}{3}\right)$ sq. units

Solution. The given curves are y = x^2 and $y = \frac{2}{1+x^2}$. Here y = x^2 is upward parabola with vertex at origin.

Also, $y = \frac{2}{1+x^2}$ is a curve symm. with respect to y-axis.

At x = 0,y = 2

$$\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2} < 0 \quad \text{for } x > 0$$

∴ Curve is decreasing on (0, ∞)

Moreover $\frac{dy}{dx} = 0$ at $x = 0$

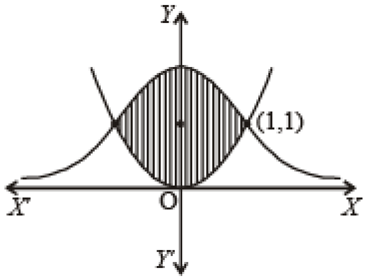
⇒ At (0,2) tangent to curve is parallel to x – axis.

As x → ∞ , y → 0

∴ y = 0 is asymptote of the given curve.

For the given curves, point of intersection : solving their equations we get x = 1, y = 1, i.e., (1,1).

Thus the graph of two curves is as follows:



$\therefore$ The required area $= 2\int_0^1\left(\frac{2}{1+x^2}-x^2\right)dx$

$=\left(4\tan^{-1}x-\frac{2x^3}{3}\right)_0^1 = 4.\frac{\pi}{4}-\frac{2}{3} = \pi-\frac{2}{3}$ sq. units.

Q. 25. Determine a positive integer n $\le$ 5, such that $\int_0^1 e^x(x-1)^n\,dx = 16-6e$ **(1992 - 4 Marks)**

Ans. n = 3

Solution. Given that $\int_0^1 e^x(x-1)^n\,dx = 16-6e$

where n ∈ N and n $\le$ 5

To find the value of n.

Let $I_n = \int_0^1 e^x(x-1)^n\,dx$

$= [(x-1)^n e^x]_0^1 - \int_0^1 n(x-1)^{n-1}e^x\,dx$

$= -(-1)^n - \int_0^1 n(x-1)^{n-1}e^x\,dx$

$I_n = (-1)^{n+1} - nI_{n-1}$ ....(1)

Also, $I_1 = \int_0^1 e^x(x-1)\,dx$

$= [e^x(x-1)]_0^1 - \int_0^1 e^x\,dx = -(-1)-(e^x)_0^1$

$= -(e-1) = 2-e$

Using eq. (1), $I_2 = (-1)^3 - 2I_1 = -1 - 2(2-e) = 2e - 5$

Similarly, $I_3 = (-1)^4 - 3I_2 = 1 - 3(2e-5) = 16 - 6e$

$\therefore n = 3$

Q. 26. Evalute $\int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} dx.$ **(1993 - 5 Marks)**

Ans. $\frac{1}{2}\log 6 - \frac{1}{10}$ **27.** $2n + 1 - \cos\gamma$

Solution.

$$I = \int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} dx$$

$$= \int_2^3 \frac{2x^5 - 2x^3 + x^4 + 2x^2 + 1}{(x^2+1)^2 (x^2-1)} dx$$

$$= \int_2^3 \frac{2x^3(x^2-1) + (x^2+1)^2}{(x^2+1)^2(x^2-1)} dx$$

$$= \int_2^3 \frac{2x^3}{(x^2+1)^2} + \int_2^3 \frac{1}{x^2-1} dx$$

$$= \int_2^3 \frac{x^2 . 2x}{(x^2+1)^2} + \left[\frac{1}{2}\log\frac{x-1}{x+1}\right]_2^3$$

$$= \int_5^{10} \frac{t-1}{t^2} dt + \frac{1}{2}\left(\log\frac{2}{4} - \log\frac{1}{3}\right)$$

Put $x^2 + 1 = t$,2 x dx = dt

When $x \to 2, t \to 5$, $x \to 3, t \to 10$

$$= \int_5^{10} \left(\frac{1}{t} - \frac{1}{t^2}\right) dt + \frac{1}{2}\log\frac{3}{2} = \left(\log|t| + \frac{1}{t}\right)_5^{10} + \frac{1}{2}\log\frac{3}{2}$$

$$= \log 10 - \log 5 + \frac{1}{10} - \frac{1}{5} + \frac{1}{2}\log\frac{3}{2}$$

$$= \log 2 + \left(-\frac{1}{10}\right) + \frac{1}{2}\log\frac{3}{2} = \frac{1}{2}\left[2\log 2 + \log\frac{3}{2}\right] - \frac{1}{10}$$

$$= \frac{1}{2}\log 6 - \frac{1}{10}$$

Q. 27. Show that $\int_0^{n\pi+v} |\sin x|\, dx = 2n + 1 - \cos v$ **where n is a positive integer and 0 $\leq$ n < π. (1994 - 4 Marks)**

Ans. 2n + 1 - cosγ

Solution. To prove that $\int_0^{n\pi+v} |\sin x|\, dx = 2n + 1 - \cos v$

$$\text{Let } I = \int_0^{n\pi+v} |\sin x|\, dx$$

$$= \int_0^{v} |\sin x|\, dx + \int_v^{n\pi+v} |\sin x|\, dx$$

Now we know that | sin x| is a periodic function of period π , So using the property..

$$= \int_a^{a+nT} f(x)dx = n\int_0^{T} f(x)dx$$

where n ∈ I and f (x) is a periodic function of period T

$$\text{We get, } I = \int_0^{v} \sin x\, dx + n\int_0^{\pi} \sin x\, dx$$

$$[\because |\sin x| = \sin x \text{ for } 0 \leq x \leq v]$$

$$= (-\cos x)_0^{v} + n(-\cos x)_0^{\pi} = -\cos v + 1 + n(1+1)$$

$$= 2n + 1 - \cos v = \text{R.H.S.}$$

Q. 28. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$? (1994 - 5 Marks)

Ans. 121 : 4

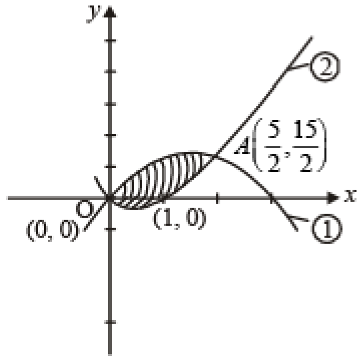
Solution. The given equations of parabola are

$y = 4x - x^2$ or $(x - 2)^2 = -(y - 4)$ ..... (1)

and $y = x^2 - x$ or $\left(x-\frac{1}{2}\right)^2 = \left(y+\frac{1}{4}\right)$ ...... (2)

Solving the equations of two parabolas we get their points of intersection as $O(0,0), A\left(\frac{5}{2},\frac{15}{4}\right)$

Here the area below x - axis,



$$A_1 = \int_0^1 (-y_2)\,dx = \int_0^1 (x - x^2)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units.}$$

Area above x - axis,

$$A_2 = \int_0^{5/2} y_1\,dx - \int_1^{5/2} y_2\,dx$$

$$= \int_0^{5/2} (4x - x^2)\,dx - \int_1^{5/2} (x^2 - x)\,dx$$

$$= \left(2x^2 - \frac{x^3}{3}\right)_0^{5/2} - \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^{5/2}$$

$$= \left(\frac{25}{2} - \frac{125}{24}\right) - \left[\left(\frac{125}{24} - \frac{25}{8}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{25}{2} - \frac{125}{12} + \frac{25}{8} - \frac{1}{6} = \frac{300 - 250 + 75 - 4}{24} = \frac{121}{24}$$

∴ Ratio of areas above x – axis and below x – axis.

$$A_2 : A_1 = \frac{121}{24} : \frac{1}{6} = \frac{121}{4} = 121 : 4$$

Q. 29. Let $I_m = \int_0^{\pi} \frac{1 - \cos mx}{1 - \cos x}\, dx$. **Use mathematical induction to prove that $I_m = m\pi$, m = 0, 1, 2, ... (1995 - 5 Marks)**

Solution. Given $I_m = \int_0^{\pi} \frac{1 - \cos mx}{1 - \cos x} dx$

To prove: $I_m = m\pi$, m= 0, 1, 2, ...

For m = 0

$$I_0 = \int_0^{\pi} \frac{1 - \cos 0}{1 - \cos x} dx = \int_0^{\pi} \frac{1 - 1}{1 - \cos x} dx = 0$$

∵ Result is true for m = 0

For m = 1,

$$I_1 = \int_0^{\pi} \frac{1 - \cos x}{1 - \cos x} dx = \int_0^{\pi} 1.dx$$

$$(x)_0^{\pi} = \pi - 0 = \pi$$

∴ Result is true for m = 1

Let the result be true for $m \le k$ i.e. $I_k = k\pi$ ....(1)

Consider $I_{k+1} = \int_0^{\pi} \frac{1 - \cos(k+1)x}{1 - \cos x} dx$

Now, 1 - cos(k+ 1)x

= 1 – cos kx cosx + sin kx sin x

= 1 + cos kx cos x + sin kx sin x – 2 cos kx cos x

= 1 + cos (k – 1) x – 2 cos kx cos x

= 2 – (1 – cos (k – 1) x) –2 cos kx cos x

= 2 (1 – cos kx cos x) – (1– cos (k – 1) x)

= 2 – 2 cos kx + 2 cos kx – 2 cos kx cos x – [1– cos (k – 1) x]

= 2 (1 – cos kx) + 2 cos kx (1– cos x) – (1 – cos (k – 1) x)

$$\therefore\ I_{k+1} = \int_0^{\pi} \frac{2(1-\cos kx) + 2\cos kx(1-\cos x) - (1-\cos(k-1)x)}{1-\cos x}\,dx$$

$$= 2\int_0^{\pi} \frac{1-\cos kx}{1-\cos x}\,dx + 2\int_0^{\pi} \cos kx\,dx - \int_0^{\pi} \frac{1-\cos(k-1)x}{1-\cos x}\,dx$$

$$= 2I_k + 2\left(\frac{\sin kx}{k}\right)_0^{\pi} - I_{k-1}$$

= 2 (kπ) + 2(0) - (k - 1)π [Using (i)]

= 2kπ- kπ+π = (k +1)π

Thus result is true for m=k + 1 as well. Therefore by the principle of mathematical induction, given statement is true for all m = 0, 1, 2, ...

Q. 30. Evaluate the definite integral : $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right) \cos^{-1}\left(\frac{2x}{1+x^2}\right) dx$ **(1995 - 5 Marks)**

Ans. $\frac{\pi}{12}\left[\pi + 3\log_e(2+\sqrt{3}) - 4\sqrt{3}\right]$

Solution.

Let $I = \int_{-1\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right) \cos^{-1}\left(\frac{2x}{1+x^2}\right) dx$

We know that $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1} x$

Also $\sin^{-1} y + \cos^{-1} y = \frac{\pi}{2}$

$\therefore$ We get $\frac{\pi}{2} - \cos^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1} x$

$\Rightarrow \cos^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{\pi}{2} - 2\tan^{-1} x$

$$\therefore I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right)\left[\frac{\pi}{2} - 2\tan^{-1} x\right] dx$$

$$= \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx - 2\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4 \tan^{-1} x}{1-x^4} dx$$

$$= 2.\frac{\pi}{2}\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx - 2\times 0$$

$= $[Using $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if f is even]

$= 0$ if f is odd

$$= \pi\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx$$

$$\therefore I = -\pi\int_0^{1/\sqrt{3}} \frac{(1-x^4)-1}{1-x^4} dx$$

$$= -\pi\int_0^{1/\sqrt{3}} 1 - \frac{1}{1-x^4} dx = -\pi\int_0^{1/\sqrt{3}} \left[1 - \frac{1}{2}\left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right)\right] dx$$

$$= -\pi\left[x - \frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1+x}{1+x}\right| + \tan^{-1} x\right)\right]_0^{1/\sqrt{3}}$$

$$= -\pi\left[\frac{1}{\sqrt{3}} - \frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}\right| - \tan^{-1}\frac{1}{\sqrt{3}}\right) - 0\right]$$

$$= -\pi\left[\frac{1}{\sqrt{3}} - \frac{1}{4}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \frac{\pi}{12}\right]$$

$$= \pi\left[\frac{\pi}{12} + \frac{1}{4}\log(2+\sqrt{3}) - \frac{\sqrt{3}}{3}\right]$$

$$= \frac{\pi}{12}[\pi + 3\log(2+\sqrt{3}) - 4\sqrt{3}]$$

Q. 31. Consider a square with vertices at (1, 1), (– 1, 1), (– 1, – 1) and (1, – 1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area. (1995 - 5

Marks)

Ans. $\frac{16\sqrt{2}-20}{3}$

Solution. Let us consider any point P (x, y) inside the square such that its distance from origin ≤ its distance from any of the edges say AD

$\therefore OP \le PM$ or $\sqrt{(x^2+y^2)} < 1-x$

or $y^2 \le -2\left(x-\frac{1}{2}\right)$ .... (1)

Above represents all points within and on the parabola 1. If we consider the edges BC then OP < PN will imply

$y^2 \le 2\left(x+\frac{1}{2}\right)$ .... (2)

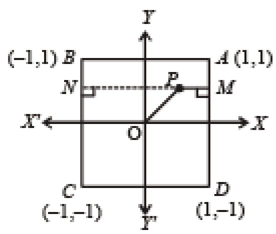
Similarly if we consider the edges AB and CD, we will have

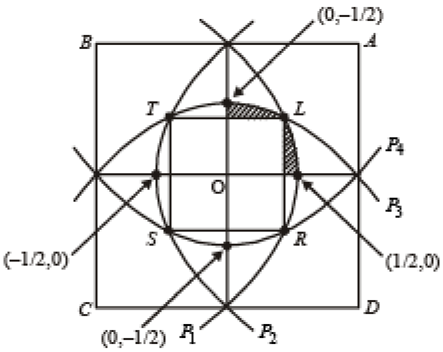
$x^2 \le -2\left(y-\frac{1}{2}\right)$ .... (3)

$x^2 \le 2\left(y+\frac{1}{2}\right)$ .... (4)

Hence S consists of the region bounded by four parabolas meeting the axes at $\left(\pm\frac{1}{2}, 0\right)$ and $\left(0, \pm\frac{1}{2}\right)$

The point L is intersection of P_1 and P_3 given by (1) and (3).





$y^2 - x^2 = -2(x - y) = 2(y-x)$ 0

$\therefore$ y - x = $\therefore$ y = x $\therefore$ $x^2 + 2x - 1 = 0 \Rightarrow (x+1)^2 = 2$

$\therefore \quad x = \sqrt{2} - 1$ as x is +ve

$\therefore \quad L$ is $(\sqrt{2}-1, \sqrt{2}-1)$

$$\therefore \text{ Total area } = 4\left[\text{square of side } (\sqrt{2}-1) + 2\int_{\sqrt{2}-1}^{1/2} y\,dx\right]$$

$$= 4\left\{(\sqrt{2}-1)^2 + 2\int_{\sqrt{2}-1}^{1/2} \sqrt{(1-2x)}\,dx\right\}$$

$$= 4\left[3 - 2\sqrt{2} - \frac{2}{2}\cdot\frac{2}{3}\{(1-2x)^{3/2}\}_{\sqrt{2}-1}^{1/2}\right]$$

$$= 4\left[3 - 2\sqrt{2} - \frac{2}{3}\{0 - (1 - 2\sqrt{2} + 2)^{3/2}\right]$$

$$= 4\left[3 - 2\sqrt{2} + \frac{2}{3}(3 - 2\sqrt{2})^{3/2}\right]$$

$$= 4(3 - 2\sqrt{2})\left[1 + \frac{2}{3}\sqrt{(3 - 2\sqrt{2})}\right]$$

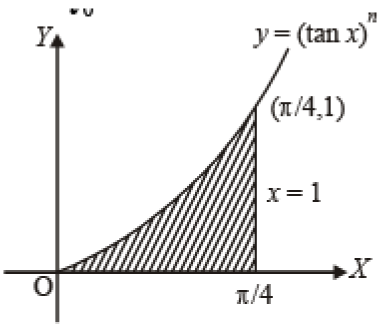
$$= 4(3 - 2\sqrt{2})\left[1 + \frac{2}{3}(\sqrt{2} - 1)\right]$$

$$= \frac{4}{3}(3 - 2\sqrt{2})(1 + 2\sqrt{2}) = \frac{4}{3}[(4\sqrt{2} - 5)] = \frac{16\sqrt{2} - 20}{3}$$

Q. 32. Let A_n be the area bounded by the curve y = (tan x)n and the lines x = 0, y = 0 and x = π/4. Prove that for n > 2, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$. **(1996 - 3 Marks)**

Solution.

We have $A_n = \int_0^{\pi/4} (\tan x)^n \, dx$



Since 0 < tan x < 1, when 0 < x < π /4 , we have

0 < $(\tan x)^{n+1}$ < (tanx)n for each n ∈ N

$$\Rightarrow \int_0^{\pi/4} (\tan x)^{n+1} dx < \int_0^{\pi/4} (\tan x)^n dx$$

$$\Rightarrow A_{n+1} < A_n$$

Now, for n > 2

$$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}] dx$$

$$= \int_0^{\pi/4} (\tan x)^n + (1 + \tan^2 x) dx$$

$$= \int_0^{\pi/4} (\tan x)^n + (\sec^2 x) dx$$

$$= \left[\frac{1}{(n+1)} (\tan x)^{n+1} \right]_0^{\pi/4}$$

$$\left[\because \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \frac{1}{(n+1)} (1 - 0)$$

Since $A_{n+2} < A_{n+1} < A_n$. we get, $A_n + A_{n+2} < 2A_n$

$\Rightarrow \quad \frac{1}{n+1} < 2A_n \Rightarrow \frac{1}{2n+2} < A_n$ ....(1)

Also for $n > 2, A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$

$\Rightarrow 2A_n < \frac{1}{n-1}$

$\Rightarrow A_n < \frac{1}{2n-2}$ ....(2)

Combining (1) and (2) we get

$\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$ Hence Proved.

Subjective Problems of Definite Integrals & Applications (Part - 3)

Q. 33. Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx.$ **(1997 - 5 Marks)**

Ans. π^2

Solution.

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx = I \quad \text{(say)}$$

$$\text{or } I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x} dx$$

$$I = 0 + 2\int_{0}^{\pi} \frac{2x\sin x}{1+\cos^2 x} dx \quad \left[\because \frac{2x}{1+\cos^2 x} \text{ is an odd function}\right]$$

$$\text{or} \quad I = 4\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x} dx \qquad .....(1)$$

$$\text{or} \quad I = 4\int_{0}^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\{\cos(\pi - x)\}^2} dx = 4\int_{0}^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$$

$$\text{or} \quad I = 4\pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} dx - \int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$$

$$\text{or} \quad I = 4\pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} dx - 1 \qquad [\text{from (1)}]$$

$$\text{or} \quad 2I = 4\pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Putting cos x = t, – sin x dx = dt

When x → 0,t →1 and when x → p,t → -1

$$\Rightarrow I = 2\pi\int_{1}^{-1}\frac{-dt}{1+t^2} = 2\pi\int_{-1}^{1}\frac{dt}{1+t^2} = 4\pi\int_{0}^{1}\frac{dt}{1+t^2}$$

$$\Rightarrow I = 4\pi\left(\tan^{-1}t\right)_0^1 = 4\pi\{\tan^{-1}(1) - \tan^{-1}(0)\}$$

$$\Rightarrow I = 4\pi\left\{\frac{\pi}{4} - 0\right\} = \pi^2$$

Q. 34. Let f(x) = Maximum {x^2, $(1 – x)^2$, 2x(1 – x)}, where 0 $\leq$ x $\leq$ 1. Determine the area of the region bounded by the curves y = f(x), x-axis, x = 0 and x = 1.

Ans. $\frac{17}{27}$ sq. units

Solution. We draw the graph of y = x^2, y = $(1– x)^2$ and y = 2x (1– x) in figure.

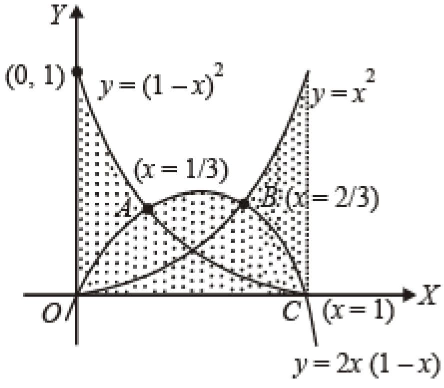
Let us find the point of intersection of y = x2 and y = 2x (1– x)

The x – coordinate of the point of intersection satisfies the equation x^2 = 2x (1– x), ⇒ $3x^2$ = 2x ⇒ 0 or x = 2 /3

∴ At B, x = 2/3

Similarly, we find the x coordinate of the points of intersection of y = $(1 – x)^2$ and y = 2x (1– x) are x = 1/3 and x = 1

∴ At A, x = 1/3 and at C x = 1



From the figure it is clear that

$$f(x)=\begin{cases}(1-x)^2 & \text{for } 0\le x\le 1/3\\ 2x(1-x) & \text{for } 1/3\le x\le 2/3\\ x^2 & \text{for } 2/3\le x\le 1\end{cases}$$

The required area A is given by

$$A=\int_0^1 f(x)dx$$

$$=\int_0^{1/3}(1-x)^2dx+\int_{1/3}^{2/3}2x(1-x)dx+\int_{2/3}^{1}x^2dx$$

$$=\left[-\frac{1}{3}(1-x)^3\right]_0^{1/3}+\left[x^2-\frac{2x^2}{3}\right]_{1/3}^{2/3}+\left[\frac{1}{3}x^3\right]_{2/3}^{1}$$

$$=-\frac{1}{3}\left(\frac{2}{3}\right)^3+\frac{1}{3}\left(\frac{2}{3}\right)^2-\frac{2}{3}\left(\frac{2}{3}\right)^3-\left(\frac{1}{3}\right)^2+\frac{2}{3}\left(\frac{1}{3}\right)^3 \quad +\frac{1}{3}(1)-\frac{1}{3}\left(\frac{2}{3}\right)^3=\frac{17}{27}\text{ sq. units}$$

Q. 35. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx=2\int_0^1 \tan^{-1}x\,dx.$ **Hence or otherwise, evaluate the integral** $\int_0^1 \tan^{-1}(1-x+x^2)\,dx.$

Ans. log 2

Solution.

$$\therefore I=\int_0^1 ydx=\int_0^1\tan^{-1}xdx-\int_0^1\tan^{-1}(x-1)dx$$

$$=\int_0^1\tan^{-1}xdx-\int_0^1\tan^{-1}\{(1-x)-1\}$$

$$\left[\text{Using}\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^1\tan^{-1}x\,dx-\int_0^1(-\tan^{-1}x)\,dx=2\int_0^1\tan^{-1}xdx\ \text{(Proved)}$$

$$=2\left[x\tan^{-1}dx-\frac{1}{2}\log(1+x^2)\right]_0^1$$

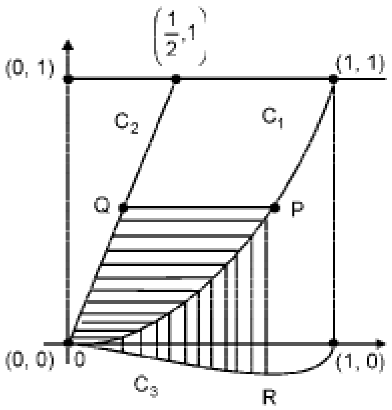
$$=\frac{\pi}{2}-\log 2 \qquad \text{...... (1)}$$

Now, $\int_0^1 \tan^{-1}(1-x+x^2)dx$

$$= \int_0^1 \cot^{-1}\frac{1}{1-x+x^2}dx = \int_0^1\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{1-x+x^2}\right)dx$$

$$= \left[\frac{\pi}{2}x\right]_0^1 - I = \frac{\pi}{2} - \left(\frac{\pi}{2} - \log 2\right) = \log 2 \text{ by (1)}$$

Q. 36. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure.) If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function f(x).



Ans. f (x) = x^3-x^2

Solution. f (x) = x^3-x^2

Let P be on C_1, $y = x^2$ be (t, t^2)

∴ Ordinate of Q is also t^2.

Now Q lies on y = 2x, and y = t2

∴ $x = t^2/2$

$\therefore \quad Q\left(\frac{t^2}{2}, t^2\right)$

For point R, x = t and it is on y = f (x)

∴ R is [t, f (t)]

$$\text{Area } OPQ = \int_0^{t^2} (\underset{C_1}{x_1} - \underset{C_2}{x_2})dy = \int_0^{t^2} \left(\sqrt{y} - \frac{y}{2}\right) dy$$

$$= \frac{2}{3}t^3 - \frac{t^4}{4}$$ ...(1)

$$\text{Area } OPR = \int_{0_{C_1}}^{t} ydx + \left|\int_{0_{C_3}}^{t} ydx\right|$$

$$= \int_0^t x^2 dx + \left|\int_0^t f(x)dx\right| = \frac{t^3}{3} + \left|\int_0^t f(x)dx\right|$$

Equating (1) and (2), we get,

$$\frac{t^3}{3} - \frac{t^4}{4}\left|\int_0^t f(x)dx\right|$$

Differentiating both sides, we get,

$t^2 - t^3 =- f(t)$

∴ $f(t) = x^3 - x^2$.

Q. 37. Integrate $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx.$

Ans. $\pi/2$

Solution.

$$I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi - x)} + e^{-\cos(\pi - x)}} \quad \Rightarrow I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}$$

$$\text{Adding, } 2I = \int_0^{\pi} dx = \pi \Rightarrow I = \pi/2$$

Q. 38. Let f(x) be a continuous function given by $f(x) = \begin{Bmatrix} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{Bmatrix}$

Ans. $\frac{257}{192}$ sq. units

Solution.

$$f(x) = \begin{cases} x^2 + ax + b; & x < -1 \\ 2x & ; -1 \le x \le 1 \\ x^2 + ax + b; & x > 1 \end{cases}$$

∵ f (x) is continuous at x = – 1 and x = 1

∴ $(-1)^2 + a(-1) + b = -2$ and $2 = (1)^2 + a \cdot 1 + b$ i.e. $a - b = 3$ and $a + b = 1$

On solving we get a = 2, b = –1

$$\therefore \quad f(x) = \begin{cases} x^2 + 2x - 1; & x < -1 \\ 2x & ; -1 \le x \le 1 \\ x^2 + 2x - 1; & x > 1 \end{cases}$$

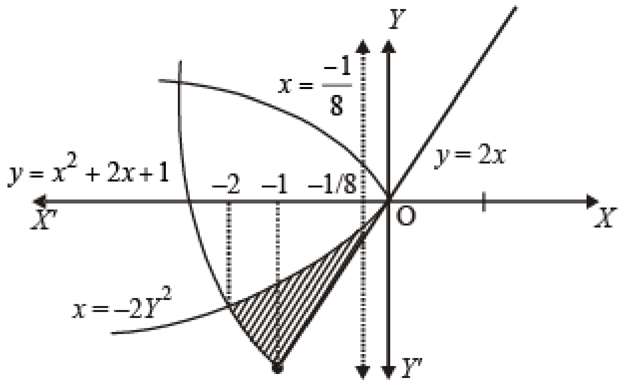
Given curves are y = f (x), $x = -2y^2$ and $8x + 1 = 0$

Solving $x = -2y^2$, $y = x^2 + 2x - 1$ $(x < -1)$ we get x = – 2

Also y = 2x, $x = -2y^2$ meet at (0, 0)

$y = 2x$ and $x = -1/8$ meet at $\left(-\frac{1}{8}, \frac{-1}{4}\right)$

The required area is the shaded region in the figure.



∴ Required area

NOTE THIS STEP :

$$=\int_{-2}^{-1}\left[\sqrt{\frac{-x}{2}}-(x^2+2x-1)\right]dx+\int_{-1}^{-1/8}\left[\sqrt{\frac{-x}{2}}-2x\right]dx$$

$$=\left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3}-\frac{x^3}{3}-x^2+x\right]_{-2}^{-1}+\left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3}-x^2\right]_{-1}^{-1/8}$$

$$=\left(\frac{\sqrt{2}}{3}+\frac{1}{3}-1-1\right)-\left(\frac{4}{3}+\frac{8}{3}-4-2\right)+\left(\frac{\sqrt{2}}{3}.\frac{1}{16\sqrt{2}}-\frac{1}{64}\right)-\left(\frac{\sqrt{2}}{3}-1\right)$$

$$=\left(\frac{\sqrt{2}-5}{3}\right)-\left(\frac{4+8-18}{3}\right)+\left(\frac{4-3}{192}\right)-\left(\frac{\sqrt{2}-3}{3}\right)$$

$$=\frac{257}{192}\text{ sq. units}$$

Q. 39. For x > 0, let $f(x)=\int_e^x\frac{\ln t}{1+t}dt$. **Find the function** $f(x)+f\left(\frac{1}{x}\right)$ $f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}$.
Here, lnt = log$_e$t.

Solution.

$f(x)=\int_1^x\frac{\ln t}{1+t}dt$ for $x>0$ (given)

Now $f\left(\frac{1}{x}\right)=\int_1^{1/x}\frac{\ln t}{1+t}dt$: Put $t=\frac{1}{u}$, so that

$dt=-\frac{1}{u^2}du$

Therefore $f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln(1/u)}{1+\frac{1}{u}} \cdot \frac{(-1)}{u^2} du$

$$= \int_1^x \frac{\ln u}{u(u+1)} du = \int_1^x \frac{\ln t}{t(t+1)} dt$$

Now, $f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln t}{1+t} dt + \int_1^x \frac{\ln t}{t(1+t)} dt$

$$= \int_1^x \frac{(1+t)\ln t}{t(1+t)} dt = \int_1^x \frac{\ln t}{t} dt = \frac{1}{2}(\ln t)^2 \Big]_1^x = \frac{1}{2}(\ln x)^2$$

Put $x = e$, hence $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}(\ln e)^2 = \frac{1}{2}$

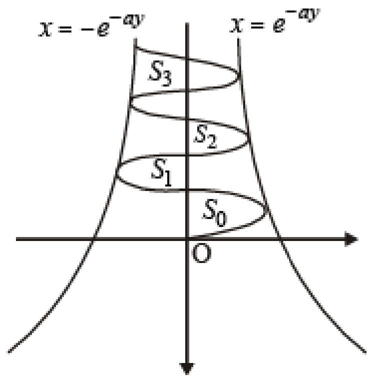
Hence Proved.

Q. 40. Let b ≠ 0 and for j = 0, 1, 2, …, n, let Sj be the area of the region bounded by the y-axis and the curve $xe^{a}y$ = sin by, $\frac{jr}{b} \le y \le \frac{(j+1)\pi}{b}$. Show th at $S_0, S_1, S_2, \ldots, S_n$ are in geometric progression. Also, find their sum for a = -1 and b = π.

Ans. $\frac{\pi(1+e)}{1+\pi^2}\left(\frac{e^{n+1}-1}{e-1}\right)$

Solution. Given that x = sin by. $e^{-ay} \Rightarrow -e^{-ay} \le x \le e^{-ay}$

The figure is drawn taking a and b both +ve. The given curve oscillates between $x = e^{-ay}$ and $x = -e^{-ay}$



Clearly, $S_j = \int_{\frac{j\pi}{b}}^{\frac{(j+1)\pi}{b}} \sin by . e^{-ay} . dy$

Integrating by parts, $I = \int \sin by . e^{-ay} dy$

We get $I = -\frac{e^{-ay}}{a^2+b^2}(a \sin by + b \cos by)$

So, $S_j = \left| -\frac{1}{a^2+b^2}\left[e^{-a\frac{(j+1)\pi}{b}}\{a \sin(j+1)\pi + b\cos(j+1)\pi - e^{\frac{-aj\pi}{b}}(a \sin j\pi + b \cos j\pi)\right] \right|$

$$\Rightarrow S_j = \left| -\frac{1}{a^2+b^2}\left[e^{-\frac{a}{b}(j+1)\pi} b(-1)^{j+1} - e^{-\frac{a}{b}j\pi} b(-1)^j \right] \right|$$

$$= \left| \frac{b.(-1)^j e^{-\frac{a}{b}j\pi}}{a^2+b^2}\left(e^{-\frac{a}{b}\pi} + 1 \right) \right| = b.\frac{e^{-\frac{a}{b}j\pi}}{a^2+b^2}\left(e^{-\frac{a}{b}\pi} + 1 \right)$$

Now, $\frac{S_j}{S_{j-1}} = \frac{e^{-\frac{a}{b}j\pi}}{e^{-\frac{a}{b}(j-1)\pi}} = e^{-\frac{a}{b}\pi} = \text{constant}$

$\Rightarrow S_o, S_1, S_2, \ldots\ldots\ldots S_j$ form a G.P.

For $a = -1$ and $b = \pi$ $\quad S_j = \frac{\pi e^j}{(1+\pi^2)}(1+e)$

$$\Rightarrow \sum_{j=0}^{n} S_j = \frac{\pi(1+e)}{(1+\pi^2)} \cdot \frac{(e^{(n+1)}-1)}{(e-1)}.$$

Q. 41. Find the area of the region bounded by the curves y = x^2, y = |2 – x^2| and y = 2, which lies to the right of the line x = 1.

Ans. $\left(\frac{20}{3} - 4\sqrt{2}\right)$ sq. units

Solution. The given curves are y = x^2 which is an upward parabola with vertex at (0, 0)

$y = |2 - x^2|$

or $y = \begin{cases} 2 - x^2 \text{ if } & -\sqrt{2} \le x \le \sqrt{2} \\ x^2 - 2 \text{ if } x < -\sqrt{2} \text{ or } x > \sqrt{2} \end{cases}$

or $x^2 = -(y-2);\ -\sqrt{2} < x < \sqrt{2}$ ....(2)

a downward parabola with vertex at (0, 2)

$x^2 = y + 2;\ x < -\sqrt{2},\ x > \sqrt{2}$ ....(3)

An upward parabola with vertex at (0, – 2)

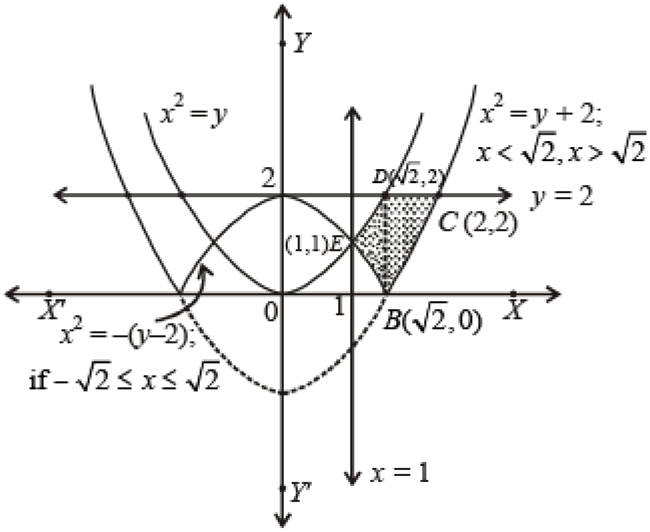
y = 2 ....(4)

A straight line parallel to x – axis

x = 1 ....(5)

A straight line parallel to y – axis

The graph of these curves is as follows.



∴ Required area = BCDEB

$$= \int_1^{\sqrt{2}} [Y_{(1)} - Y_{(2)}]\,dx + \int_{\sqrt{2}}^{2} [Y_{(4)} - Y_{(3)}]\,dx \qquad \text{..........(1)}$$

$$= \int_1^{\sqrt{2}} [x^2 - (2 - x^2)]\,dx + \int_{\sqrt{2}}^{2} [2 - (x^2 - 2)]\,dx$$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)\,dx + \int_{\sqrt{2}}^{2} (4 - x^2)\,dx$$

$$= \left[\frac{2x^3}{3} - 2x\right]_1^{\sqrt{2}} + \left[4x - \frac{x^3}{3}\right]_{\sqrt{2}}^{2}$$

$$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{2}{3} + 2\right) + \left(8 - \frac{8}{3} - 4\sqrt{2} + \frac{2\sqrt{2}}{3}\right)$$

$$= -\frac{2}{3}\sqrt{2} + \frac{4}{3} + \frac{16}{3} - \frac{10\sqrt{2}}{3}$$

$$= \frac{20 - 12\sqrt{2}}{3} = \left(\frac{20}{3} - 4\sqrt{2}\right) \text{ sq. units.}$$

Q. 42. If f is an even function then prove that

$$\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx.$$

Solution. Given that f (x) is an even function, then to prove

$$\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$$

$$\text{Let } I = \int_0^{\pi/2} f(\cos 2x)\cos x\,dx \qquad \text{.........(1)}$$

$$= \int_0^{\pi/2} f\left[\cos 2\left(\frac{\pi}{2} - x\right)\right]\cos\left(\frac{\pi}{2} - x\right)dx$$

$$\left[\text{Using } \int_0^a f(x)dx = \int_0^a f(a - x)dx\right]$$

$$= \int_0^{\pi/2} f(-\cos 2x)\sin x\,dx$$

$$I = \int_0^{\pi/2} f(\cos 2x)\sin x\,dx \qquad \text{.......(2)}$$

[As f is an even function] Adding two values of I in (1) and (2) we get

$$2I=\int_0^{\pi/2} f(\cos 2x)(\sin x+\cos x)dx$$

$$\Rightarrow I=\frac{\sqrt{2}}{2}\int_0^{\pi/2} f(\cos 2x)\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right]dx$$

$$I=\frac{\sqrt{2}}{2}\int_0^{\pi/2} f(\cos 2x)\cos(x-\pi/4)dx$$

Let $x - \pi/4 = t$ so that $dx = dt$

as $x \to 0$, $t \to -\pi/4$ and as $x \to \pi/4$, $t \to \pi/2-\pi/4 = \pi/4$

$$\therefore \quad I=\frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f[\cos 2(t+\pi/4)]\cos t\, dx$$

$$=\frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f[-\sin 2t]\cos t\, dt$$

$$=\frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f(\sin 2t)\cos t\, dt$$

$$=\frac{2}{\sqrt{2}}\int_0^{\pi/4} f(\sin 2t)\cos t\, dt$$

$$=\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx$$

R.H.S. Hence proved.

Q. 43. If $y(x)=\int_{\pi^2/16}^{x^2}\frac{\cos x\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$, then find $\frac{dy}{dx}$ at $x=\pi$

Ans. 2π

Solution. We have,

$$y(x)=\int_{\pi^2/16}^{x^2}\frac{\cos x\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

$$=\cos x\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

[$\because$ cos x is independent of θ]

$$\Rightarrow \frac{dy}{dx} = -\sin x \int_{\pi^2/16}^{x^2} \left[\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} \right] d\theta \quad + \cos x \frac{d}{dx}\left[\int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta \right]$$

$$= -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta \quad + \cos x \left[\int_{\pi^2/16}^{x^2} \frac{\cos x}{1+\sin^2 x} .2x - 0 \right]$$ (Using Leibnitz thm.)

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\pi} = -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \frac{(\cos^2\pi).2\pi}{1+\sin^2\pi}$$

$$= 0 + 2\pi = 2\pi$$

Q. 44. Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$

Ans. $\frac{4\pi}{\sqrt{3}}\left[\tan^{-1} 3 - \frac{\pi}{4}\right]$

Solution.

Let $I = \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$

$$= \int_{-\pi/3}^{\pi/3} \frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$$

The second integral becomes zero integrand being an odd function of x.

$$= 2\pi \int_0^{\pi/3} \frac{dx}{2 - \cos\left(x + \frac{\pi}{3}\right)}$$

{using the prop. of even function and also $|x| = x$ for $0 \le x \le \pi/3$}

Let $x + \pi/3 = y \Rightarrow dx = dy$

also as $x \to 0, y \to \pi/3$ as $x \to \pi/3$, $y \to 2\pi/3$

∴ The given integral becomes

$$= 2\pi\int_{\pi/3}^{2\pi/3}\frac{dy}{2-\cos y} = 2\pi\int_{\pi/3}^{2\pi/3}\frac{dy}{2-\frac{1-\tan^2 y/2}{1+\tan^2 y/2}}$$

$$= 2\pi\int_{\pi/3}^{2\pi/3}\frac{\sec^2 y/2}{3\tan^2 y/2+1}dy$$

$$= \frac{2\pi}{3}\int_{\pi/3}^{2\pi/3}\frac{\sec^2 y/2}{\tan^2 y/2+(1/\sqrt{3})^2}dy$$

$$= \frac{4\pi\sqrt{3}}{3}\left[\tan^{-1}(\sqrt{3}\tan y/\sqrt{2})\right]_{\pi/3}^{2\pi/3}$$

$$= \frac{4\pi}{3}\left[\tan^{-1}3-\tan^{-1}1\right] = \frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3-\pi/4\right]$$

Q. 45. Evaluate $\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$

Ans. $\frac{24}{5}\left[e\cos\left(\frac{1}{2}\right)+\frac{1}{2}e\sin\left(\frac{1}{2}\right)-1\right]$

Solution. Let

$$I = \int_0^{\pi} e^{|\cos x|}\left[2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx$$

$$= \int_0^{\pi} e^{|\cos x|}2\sin\left(\frac{1}{2}\cos x\right)\sin x\,dx + \int_0^{\pi} e^{|\cos x|}3\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

$= I_1+I_2$

Now using the property that

$$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ if } f(2a-x)=f(x)$$
$$=0 \text{ if } f(2a-x)=-f(x)$$

We get, $I_1 = 0$

and $I_2 = 2\int_0^{\pi/2} e^{|\cos x|}3\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$

$$= 6\int_0^{\pi/2} e^{\cos x}\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$,

$\therefore \quad I_2 = 6\int_0^1 e^t \cos t/2\,dt$

Integrating by parts, we get

$$I_2 = 6[(e^t \cos t/2)_0^1 + \frac{1}{2}\int_0^1 e^t \sin t/2\, dt]$$

$$= 6\left\{e\cos(1/2) - 1 + \frac{1}{2}\left\{(e^t \sin t/2)_0^1 - \frac{1}{2}\int_0^1 e^t \cos t/2\, dt\right\}\right]$$

$$I_2 = 6\left[e\cos\left(\frac{1}{2}\right) - 1 + \frac{1}{2}\left\{e\sin\left(\frac{1}{2}\right) - \frac{1}{2}\cdot\frac{1}{6}I_2\right\}\right]$$

$$I_2 = 6\left[e\cos(1/2) - 1 + \frac{1}{2}(e\sin t/2) - \frac{1}{24}I_2\right]$$

$$I_2 + \frac{1}{4}I_2 = 6\left[e\cos(1/2) + \frac{1}{2}e\sin(1/2) - 1\right]$$

$$\frac{5I_2}{4} = 6\left[e\cos(1/2) + \frac{1}{2}e\sin(1/2) - 1\right]$$

$$\Rightarrow I_2 = \frac{24}{5}\left[e\cos(1/2 + \frac{1}{2}e\sin\left(\frac{1}{2}\right) - 1\right]$$

Q. 46. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$.

Ans. $\frac{1}{3}$ sq. units

Solution. The given curves are, $x^2 = y$ ....(i)

$x^2 = - y$ ....(ii)

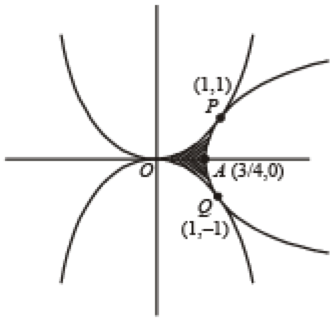
$y^2 = 4x - 3$ ....(iii)

Clearly point of intersection of (i) and (ii) is (0, 0). For point of intersection of (i) and (iii), solving them as follows

$x^4 - 4x + 3 = 0$ $(x-1)(x^3 + x^2 + x - 3) = 0$

|or (x - 1)2 (x2 + 2x + 3)= 0 ; ⇒ x = 1 and then y = 1

∴ Req. point is (1, 1). Similarly point of intersection of (ii) and (iii) is (1, – 1). The graph of three curves is as follows:



We also observe that at x = 1 and y = 1

$\frac{dy}{dx}$ for (i) and (iii) is same and hence the two curves touch each other at (1, 1).
Same is the case with (ii) and (iii) at (1, –1).

Required area = Shaded region in figure = 2 (Ar OPA)

$$= 2\left[\int_0^1 x^2 dx - \int_{3/4}^1 \sqrt{4x-3}\, dx\right]$$

$$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{2(4x-3)^{3/2}}{4\times 3}\right)_{3/4}^1\right] = 2\left[\frac{1}{3} - \frac{1}{6}\right]$$

$$= 2\times\frac{1}{6} = \frac{1}{3} \text{ sq. units}$$

Q. 47. f(x) is a differentiable function an d g(x) is a double differentiable function such that |f(x)| ≤ 1 and f '(x) = g(x). If $f^2(0) + g^2(0) = 9$. Prove that there exists some c∈ (-3, 3) such that g (c).g ''(c) < 0 .

Solution. Given that f (x) is a differentiable function such that f'(x) = g (x), then

$$\int_0^3 g(x)\,dx = \int_0^3 f'(x)dx = [f(x)]_0^3 = f(3) - f(0)$$

But $|f(x)| < 1 \Rightarrow -1 < f(x) < 1, \forall x \in R$

$\therefore \quad f(3) = f(0) \in (-1, 1)$

Similarly

$$\int_{-3}^{0} g(x)dx = \int_{-3}^{0} f'(x)dx = [f(0) - f(3)] \in (-2, 2)$$

Also given $[f(0)]^2 + [g(0)]^2 = 9$

$\Rightarrow [g(0)]^2 = 9 - [f(0)]^2$

$\Rightarrow |g(0)|^2 > 9 - 1 \qquad [\because |f(x)| < 1]$

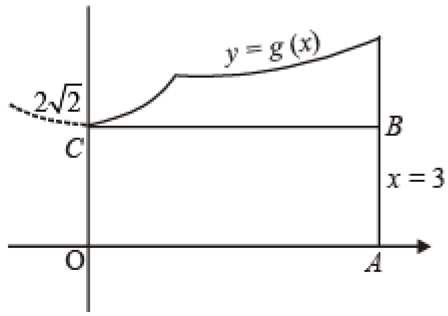
$\Rightarrow |g(0)| > 2\sqrt{2} \Rightarrow g(0) > 2\sqrt{2}$ or $g(0) < -2\sqrt{2}$

First let us consider g (0) > $2\sqrt{2}$

Let us suppose that g'' (x) be positive for all x ∈ (–3, 3).

Then g'' (x) > 0 ⇒ the curve y = g (x) is open upwards.

Now one of the two situations are possible. (i) g(x) is increasing



$\therefore \left|\int_0^3 g(x)dx\right| >$ area of rect. $OABC$

i.e. $\left|\int_0^3 g(x)dx\right| > 6\sqrt{2} > 2$

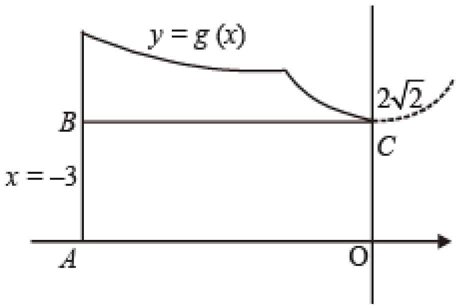
a contradiction as $\int_0^1 g(x)dx \in (-2, 2)$

∴ at least at one of the point c ∈ (–3,3), g'' (x) < 0.

But g (x) > 0 on (– 3, 3)

Hence g(x) g''(x) < 0 at some x ∈ (– 3, 3).

(ii) g (x) is decreasing



$\therefore \left|\int_{-3}^{0} g(x)dx\right| >$ area of rect. $OABC$

i.e. $\left|\int_{-3}^{0} g(x)dx\right| > 3.2\sqrt{2} = 6\sqrt{2} > 2$

a contradiction as $\int_{-3}^{0} g(x)dx \in (-2,2)$

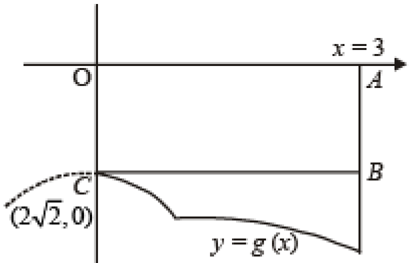
∴ at least at one of point c ∈ (-3, 3) g "(x) should be – ve. But g(x) > 0 on (–3 , 3).

Hence g (x) g" (x) < 0 at some x ∈ (–3 , 3).

Secondly let us consider g (0) < $2\sqrt{2}$.

Let us suppose that g" (x) be – ve on (– 3 , 3). then g" (x) < 0 ⇒ the curve y = g(x) is open downward.

Again one of the two situations are possible (i) g (x) is decreasing then



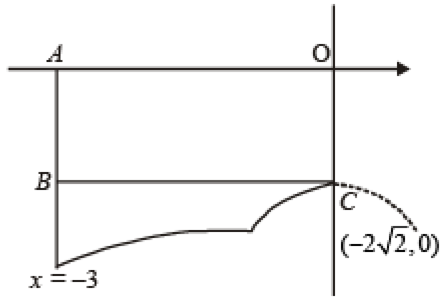
$\left|\int_{0}^{3} g(x)dx\right| >$ Ar or rect. $OABC = 3.2\sqrt{2} = 6\sqrt{2} > 2$

a contradiction as $\int_{0}^{3} g(x)dx \in (-2,2)$

∴ At least at one of the point c ∈ (– 3, 3), g" (x) is + ve. But g (x) < 0 on (– 3, 3).

Hence g(x) g" (x) < 0 for some x ∈ (– 3, 3).

(ii) g (x) is increasing then



$$\left|\int_{-3}^{0} g(x)dx\right| > \text{Ar of rect. } OABC = 3.2\sqrt{2} = 6\sqrt{2} > 2$$

a contradiction as as $\int_{-3}^{0} g(x)dx \in (-2, 2)$

∴ At least at one of the point c ∈ (– 3, 3) g" (x) is + ve.

But g (x) < 0 on (–3, 3).

Hence g (x) g" (x) < 0 for some x ∈ (– 3,3).

Combining all the cases, discussed above, we can conclude that at least at one point in (–
3, 3), g (x) g"(x) < 0.

Q. 48. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, $f(x)$ **is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of y = f(x) with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by f(x) and chord AB.**

Ans. $\frac{125}{3}$ sq.units

Solution. We have, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$

$\Rightarrow 4a^2 f(-1) + 4af(1) + f(2) = 3a^2 + 3a$

$4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3a$

$4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c$

Consider the equation

$4x^2 f(-1) + 4xf(1) + f(2) = 3x^2 + 3x$ or

$[4f(-1) - 3]x^2 + [4f(1) - 3]x + f(2) = 0$

Then clearly this eqn. is satisfied by x = a,b,c

A quadratic eqn. satisfied by more than two values of x means it is an identity and hence

$4f(-1) - 3 = 0 \Rightarrow \quad f(-1) = 3/4$
$4f(1) - 3 = 0 \quad f(1) = 3/4$
$f(2) = 0 \quad f(2) = 0$

Let $f(x) = px^2 + qx + r$ [f (x) being a quadratic eqn.]

$f(-1) = \frac{3}{4} \Rightarrow p - q + r = \frac{3}{4}$

$f(1) = \frac{3}{4} \Rightarrow p + q + r = \frac{3}{4}$

$f(2) = 0 \Rightarrow 4p + 2q + r = 0$

Solving the above we get $q = 0,\ p = -\frac{1}{4}, r = 1$

$\therefore \quad f(x) = -\frac{1}{4}x^2 + 1$

It's maximum value occur at $f'(x) = 0$ i.e., $x = 0$ then $f(x) = 1$, $\therefore$ V (0, 1)

Let A (–2, 0) be the point where curve meet x –axis.

Let B be the point $\left(h.\frac{4-h^2}{4}\right)$

As $\angle AVB = 90^0$, $m_{AV} \times m_{BV} = -1$

$$\Rightarrow \left(\frac{0-1}{-2-1}\right) \times \left(\frac{\frac{4-h^2}{4}-1}{h-0}\right) = -1$$

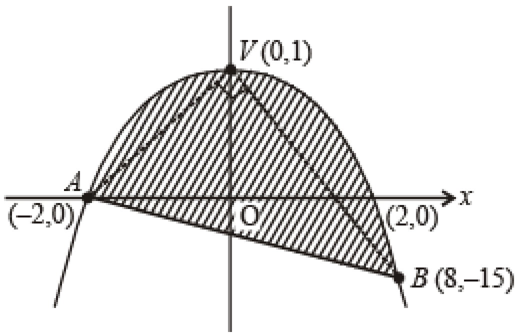
$$\Rightarrow \frac{1}{2} \times \left(\frac{-h}{4}\right) = -1 \Rightarrow h = 8$$

$\therefore$ $B(8,-15)$

Equation of chord AB is

$$y+15 = \frac{0-(-15)}{-2-8}(x-8) \Rightarrow y+15 = -\frac{3}{2}(x-8)$$

$$\Rightarrow 2y+30 = -3x+24 \Rightarrow 3y+2y+6 = 0$$



Required area is the area of shaded region given by

$$= \int_{-2}^{2}\left(-\frac{x^2}{4}+1\right)dx + \int_{-2}^{8}\left\{-\left(\frac{-6-3x}{2}\right)\right\}dx - \int_{2}^{8}\left\{-\left(-\frac{x^2}{4}+1\right)\right\}dx$$

$$= 2\int_{0}^{2}\left(-\frac{x^2}{4}+1\right)dx + \frac{1}{2}\int_{-2}^{8}(6+3x)\,dx + \frac{1}{4}\int_{2}^{8}\left(-x^2+4\right)dx$$

$$= 2\left[\left(\frac{-x^3}{12}+x\right)_0^2\right] + \frac{1}{2}\left[6x+\frac{3x^2}{2}\right]_{-2}^{8} + \frac{1}{4}\left[\frac{-x^3}{3}+4x\right]_2^8$$

$$= 2\left[\frac{-8}{12}+2\right] + \frac{1}{2}[(48+3\times32)-(-12+6)] \quad + \left[\frac{1}{4}\left(\frac{-512}{3}+32\right)-\left(\frac{-8}{3}+8\right)\right]$$

$$= 2\left[\frac{4}{3}\right] + \frac{1}{2}[150] + \frac{1}{4}\left[\frac{-432}{3}\right] = \frac{125}{3} \text{ sq. units.}$$

The value of $5050\dfrac{\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$ is.

Q. 49.

Solution.

Let $I = \int_0^1 (1-x^{50})^{100}\,dx$ and $I' = \int_0^1 (1-x^{50})^{101}\,dx$

Then, $I' = \int_0^1 1.(1-x^{50})^{101}\,dx$

$$= \left[x(1-x^{50})^{101}\right]_0^1 + 101\int_0^1 50x^{50}(1-x^{50})^{100}\,dx$$

$$= +5050\int_0^1 x^{50}(1-x^{50})^{100}\,dx$$

$$-I' = +5050\int_0^1 -x^{50}(1-x^{50})^{100}\,dx$$

$$\Rightarrow 5050\ I - I' = 5050\int_0^1 (1-x^{50})^{100}\,dx$$

$$+5050\int_0^1 \left[-x^{50}(1-x^{50})^{100}\right]dx$$

$$\Rightarrow 5050\int_0^1 (1-x^{50})^{101}\,dx = 5050\ I'$$

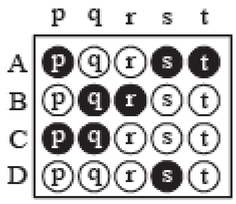
$$\Rightarrow 5050\ I = 5051\ I' \Rightarrow 5050\ \frac{I}{I'} = 5051$$

Match the following Question of Definite Integrals & Applications

Match the Following

DIRECTIONS (Q. 1 and 2) : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II.

The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :
If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



Q. 1. Column I **Column II**

(A) $\int_0^{\pi/2} (\sin x)^{\cos x}\left(\cos x \cot x - \log(\sin x)^{\sin x}\right)dx$ **(p) 1**

(B) Area bounded by $-4y^2 = x$ and $x - 1 = -5y^2$ **(q) 0**

(C) Cosine of the angle of intersection of curves y = 3x – 1 log x and y = xx – 1is **(r) 6 ln 2**

(D) Let $\frac{dy}{dx} = \frac{6}{x+y}$ where y(0) = 0 then value of y when x + y = 6 is **(s) 4/3**

Ans. (A) - p ; (B) -s ; (C) - p ; (D) – r

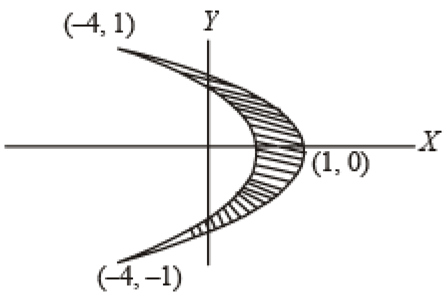
Solution.

(A) $\int_0^{\pi/2} (\sin x)^{\cos x}(\cos x \cot x - \log(\sin x)^{\sin x})dx$

$= \int_0^1 du$ where $(\sin x)^{\cos x} = u = 1$

(A) $\rightarrow$ (p)

(B)



Solving $y^2 = -\frac{1}{4}x$ and $y^2 = -\frac{1}{5}(x-1)$, we get intersection points as (– 4, + 1)

∴ Required area

$= \int_{-1}^{1} [(1-5y^2)+4y^2]dy = 2\int_0^1 (1-y^2)dy = \frac{4}{3}$,

(B) $\rightarrow$ (s)

(C) By inspection, the point of intersection of two curves y = 3x–1 log x and y = x^x – 1 is (1, 0)

For first curve $\frac{dy}{dx} = \frac{3^{x-1}}{x} + 3^{x-1} \log 3 \log x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1 = m_1$

For second curve $\frac{dy}{dx} = x^x(1+\log x)$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1 = m_2$

∵ $m_1 = m_2$ ⇒ Two curves touch each other

⇒ Angle between them is 0°

∴ cos θ = 1,

(C) ®→ (p)

(D) $\frac{dy}{dx} = \frac{6}{x+y} \Rightarrow \frac{dx}{dy} - \frac{1}{6}x = \frac{y}{6}$

I.F. = $e^{-y/6}$

⇒ Solution is $xe^{-y/6} = -ye^{-y/6} - 6e^{-y/6} + c$

⇒ $x + y + 6 = ce^{y/6}$

⇒ x + y + 6 = 6ey/6 ∴ (y (0) = 0)

⇒ $12 = 6e^{y/6}$ (using x + y = 6)

⇒ y = 6 ln 2 (D) ⟶ (r)

Q. 2. Match the integrals in Column I with the values in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column I	*Column II*
(A) $\int_{-1}^{1} \frac{dx}{1+x^2}$	(p) $\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B) $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$	(q) $2\log\left(\frac{2}{3}\right)$
(C) $\int_{2}^{3} \frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(D) $\int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

Ans. (A) - s ; (B) -s ; (C) - p ; (D) - r

Solution.

(A) $\int_{-1}^{1} \frac{dx}{1+x^2} = [\tan^{-1} x]_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1)$

$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}$

(B) $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_{0}^{1} = \sin^{-1}(1) - \sin^{-1}(0)$

$$=\frac{\pi}{2}-0=\frac{\pi}{2}$$

(C) $\int_2^3 \frac{dx}{1-x^2}=\left[\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\right]_2^3=\frac{1}{2}[\log 2-\log 3]$

$$=\frac{1}{2}\log 2/3$$

(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}=\left[\sec^{-1}x\right]_1^2=\sec^{-1}2-\sec^{-1}1$

$$=\frac{\pi}{3}-0=\frac{\pi}{3}$$

Q. 3. DIRECTIONS (Q. 3) : Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

List - I	List - II
P. The number of polynomials f (x) with non-negative integer coefficients of degree ≤ 2, satisfying f (0) = 0 and $\int_0^1 f(x)\,dx=1$, is	**1. 8**
Q. The number of points in the interval $\left[-\sqrt{13},\sqrt{13}\right]$ **at which f (x) = sin(x2) + cos(x2) attains its maximum value, is**	**2. 2**
R. $\int_{-2}^{2}\frac{3x^2}{\left(1+e^x\right)}dx$ equals	**3. 4**
S. $\frac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\int_{0}^{\frac{1}{2}}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx\right)}$	**4. 0**

	P	Q	R	S		P	Q	R	S
(a)	3	2	4	1	(b)	2	3	4	1
(c)	3	2	1	4	(d)	2	3	1	4

Ans. (d)

Solution. P(2) Let f (x) = $ax^2 + bx + c$

where a, b, c ≥ 0 and a, b, c are integers.

$\because$ f (0) = 0 $\Rightarrow$ c = 0

$\therefore$f (x) = $ax^2 + bx$

$$\text{Also} \int_0^1 f(x)dx = 1$$

$$\Rightarrow \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^1 = 1 \Rightarrow \frac{a}{3} + \frac{b}{2} = 1 \Rightarrow 2a + 3b = 6$$

Q$\because$ a and b are integers

a = 0 and b = 2

or a = 3 and b = 0

$\therefore$ There are only 2 solutions.

$$Q(3)\ f(x) = \sin x^2 + \cos x^2$$

$$f(x) \text{ is max. } \sqrt{2} \text{ at } x^2 = \frac{\pi}{4} \text{ or } \frac{9\pi}{4}$$

$$\Rightarrow x = \pm\frac{\sqrt{\pi}}{2} \text{ or } \pm\frac{3\sqrt{\pi}}{2} \in \left[-\sqrt{13}, \sqrt{13}\right]$$

$\therefore$ There are four points.

$$R(1)\ I = \int_{-2}^{2} \frac{3x^2}{1+e^x}dx = \int_{-2}^{2} \frac{3x^2}{1+e^{-x}}dx$$

$$\left[\text{Using} \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$$= \int_{-2}^{2} \frac{3x^2 e^x}{1+e^x} dx$$

$$2I = \int_{-2}^{2} \frac{3x^2 \left(1+e^x\right)}{1+e^x} dx = \int_{-2}^{2} 3x^2 dx$$

$$2I = \left(x^3\right)_{-2}^{2} = 8-(-8) = 16 \Rightarrow I = 8$$

$$S(4) \; \frac{\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx}{\int_{0}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx} = 0$$

$\because$ Numerator = 0, function being odd.

Hence option (d) is correct sequence.

Integer Value of Definite Integrals & Applications of Integrals

Q. 1. Let f: R → R be a continuous function which satisfies (2009)

$f(x)=\int_0^x f(t)\,dt.$

Then the value of f (ln 5) is (2009)

Ans. 0

Solution.

Given that $f(x)=\int_0^x f(t)dt$

Clearly $f(0)=0$. Also $f'(x)=f(x) \Rightarrow \dfrac{f'(x)}{f(x)}=1$

Integrating both sides with respect to x, we get

$$\int \frac{f'(x)}{f(x)}dx=\int 1\,dx$$

$$\Rightarrow \ln f(x)=x+\ln C \quad \Rightarrow f(x)=Ce^x$$

$$\text{Now } f(0)=0 \Rightarrow Ce^x=0 \Rightarrow C=0$$

$$\therefore \quad f(x)=0\ \forall\ x \Rightarrow \quad f(\ln 5)=0$$

Q. 2. For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by

$$f(x)=\begin{cases} x-[x] & \text{if } [x] \text{ is odd,} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$$

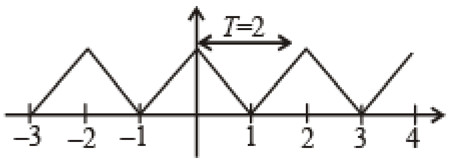
Then the value of $\dfrac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\,dx$ is **(2010)**

Ans. 4

Solution.

Given function is $f(x)=\begin{cases} x-[x] & \text{if } [x] \text{ is odd} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$

The graph of this function is as below



Clearly f(x) is periodic with period 2

Also cos πx is periodic with period 2

∴ f (x) cosπx is periodic with period 2

$$\therefore\ I=\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$$

$$=\frac{\pi^2}{10}\times 10\int_0^2 f(x)\cos \pi x\, dx$$

$$=\pi^2\left[\int_0^1 (1-x)\cos \pi x\, dx+\int_1^2 (x-1)\cos \pi x\, dx\right]$$

$$=\pi^2\left[\left\{(1-x)\frac{\sin \pi x}{\pi}\Big|_0^1+\int_0^1 \frac{\sin \pi x}{\pi}dx\right\}+\right.$$

$$\left.\left\{(x-1)\frac{\sin \pi x}{\pi}\Big|_1^2-\int_1^2 \frac{\sin \pi x}{\pi}dx\right\}\right]$$

$$=\pi^2\left[\left(-\frac{1}{\pi^2}\cos \pi x\right)\Big|_0^1-\left(-\frac{1}{\pi^2}\cos \pi x\right)^2\right]$$

$$=\left[(-\cos\pi+\cos 0)-(-\cos 2\pi+\cos\pi)\right]=\left[2+2\right]=4$$

Q. 3. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$ is **(JEE Adv. 2014)**

Ans. 2

Solution.

$$\int_0^1 4x^3\left[\frac{d^2}{dx^2}\left(1-x^2\right)^5\right]dx$$

$$= 4x^3\left[\frac{d}{dx}\left(1-x^2\right)^5\right]\Big|_0^1 - \int_0^1\left[\frac{d}{dx}\left(1-x^2\right)^5\right].12x^2dx$$

$$= -12x^2\left(1-x^2\right)^5\Big|_0^1 + \int_0^1\left(1-x^2\right)^5.24xdx$$

$$= -12\int_0^1\left(1-x^2\right)^5.(-2x)dx$$

$$= -12\left(\frac{\left(1-x^2\right)^6}{6}\right)_0^1 = -12\left(0-\frac{1}{6}\right) = 2$$

Q. 4. Let f : R → R be a function defined by $f(x) = \begin{cases}[x], & x \le 2\\ 0, & x > 2\end{cases}$ **where [x] is the greatest integer less than or equal to x, if** $I = \int_{-1}^{2}\frac{xf(x^2)}{2+f(x+1)}dx$, **, then the value of (4I – 1) is**

(JEE Adv. 2015)

Ans. 0

Solution.

$$I = \int_{-1}^{2}\frac{xf(x^2)}{2+f(x+1)}dx$$

$$-1 < x < 2 \Rightarrow 0 < x^2 < 4$$

Also $0 < x^2 < 1 \Rightarrow f(x^2) = [x^2] = 0$

$1 < x^2 < 2 \Rightarrow f(x^2) = [x2] = 1$

$2 < x^2 < 3 \Rightarrow f(x^2) = 0$ (using definition of f)

$3 < x^2 < 4 \Rightarrow f(x^2) = 0$ (using definition of f)

Also $1 \le x^2 < 2 \Rightarrow 1 \le x < \sqrt{2}$

$\Rightarrow \quad 2 \le x+1 < \sqrt{2}+1$

$\Rightarrow \quad f(x+1) = 0$

$$\therefore \quad I = \int_1^{\sqrt{2}} \frac{x \times 1}{2+0} dx = \left[\frac{x^2}{4}\right]_1^{\sqrt{2}} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$$
$$\Rightarrow \quad 4I = 1 \text{ or } 4I - 1 = 0$$

Q. 5. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t(dt)$ **for all x ∈ R and** $f:\left[0, \frac{1}{2}\right] \to [0, \infty)$ be **a continuous function. F or** $a \in \left[0, \frac{1}{2}\right]$, if $F'(a)+2$ **is the area of the region bounded by x = 0, y = 0, y = f(x) and x = a, then f(0) is** **(JEE Adv. 2015)**

Ans. 3

Solution.

$$F(x) = \int_x^{x^2+\pi/6} 2\cos^2 t \, dt$$
$$F'(\alpha) = 2\cos^2\left(\alpha^2 + \frac{\pi}{6}\right).2\alpha - 2\cos^2\alpha$$

$$F'(\alpha) + 2 = \int_0^{\alpha} f(x)dx$$
$$\Rightarrow \quad F''(\alpha) = f(\alpha)$$
$$\therefore \quad f(\alpha) = 4\alpha.2\cos\left(\alpha^2 + \frac{\pi}{6}\right).\left[-\sin\left(\alpha^2 + \frac{\pi}{6}\right)\right].2\alpha$$
$$+ 4\cos^2\left(\alpha^2 + \frac{\pi}{6}\right) - 4\cos\alpha\,(-\sin\alpha)$$

$$\therefore \quad f(0) = 4\cos^2\frac{\pi}{6} = 4 \times \frac{3}{4} = 3$$

Q. 6. If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x}) \left(\frac{12+9x^2}{1+x^2}\right) dx$ **Where $\tan^{-1}x$ takes only principal values, then the value of** $\left(\log_e |1+\alpha| - \frac{3\pi}{4}\right)$ **is** **(JEE Adv. 2015)**

Ans. 9

Solution.

$$\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$$

$$\text{Let } 9x + 3\tan^{-1}x = t \Rightarrow \frac{12+9x^2}{1+x^2}dx = dt$$

$$\therefore \quad \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$$

$$\therefore \quad \log_e\left|1+e^{9+\frac{3\pi}{4}}-1\right| - \frac{3\pi}{4} = 9$$

Q. 7. Let $f: \mathbb{R} \to \mathbb{R}$ **be a continuous odd function, which vanishes exactly at one point and f (1) = 1/2. Suppose that** $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ $\int_{-1}^{x} t\,|f(f(t))|\,dt$ for all $x \in [-1, 2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, **then the value of** $f\left(\frac{1}{2}\right)$ is

(JEE Adv. 2015)

Ans. 7

Solution.

$$\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14} \Rightarrow \lim_{x\to 1}\frac{\int_{-1}^{x} f(t)dt}{\int_{-1}^{x} t|f(f(t))|dt}$$

$$\int_{-1}^{1} f(t)dt = 0 \text{ and } \int_{-1}^{1} t|f(f(t))|dt = 0$$

f(t) being odd function

∴ Using L Hospital's rule, we get

$$\lim_{x\to 1}\frac{f(x)}{x|f(f(x))|}=\frac{1}{14}$$

$$\Rightarrow \frac{f(1)}{|f(f(1))|}=\frac{1}{14}\Rightarrow\frac{1/2}{\left|f\left(\frac{1}{2}\right)\right|}=\frac{1}{14}$$

$$\Rightarrow \left|f\left(\frac{1}{2}\right)\right|=7\Rightarrow f\left(\frac{1}{2}\right)=7$$

Q. 8. The total number of distinct x ∈ [0, 1] for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x-1$ is **(JEE Adv. 2016)**

Ans. 1

Solution.

$$\text{Let } f(x)=\int_0^x \frac{t^2}{1+t^4}dt-2x+1$$

$$\Rightarrow f'(x)=\frac{x^2}{1+x^4}-2<0\ \forall\ x\in[0,1]$$

∴ f is decreasing on [0, 1]

Also f (0) = 1

$$\text{and } f(1)=\int_0^1\frac{t^2}{1+t^4}dt-1$$

$$\text{For } 0\le t\le 1\Rightarrow 0\le\frac{t^2}{1+t^4}<\frac{1}{2}$$

$$\therefore \int_0^1\frac{t^2}{1+t^4}dt<\frac{1}{2}$$

$$\Rightarrow f(1)<0$$

∴ f(x) crosses x-axis exactly once in [0, 1]

∴ f(x) = 0 has exactly one root in [0, 1]